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7701 Lecture 29

Comments on midterm:

- overall very good — class average 88/100! (lots of bonus)
- danger line is <75 — please talk to me
- common trouble spots
  - Prob 4 3-D Fourier transforms

$$\boxed{\tilde{F}(\vec{k}) = \frac{A_0}{k^2+m^2}}$$

- $r \equiv |\vec{x}|$ or $k \equiv |\vec{k}|$ are always non-negative for real
  $\Rightarrow$ it doesn't make sense to consider $r<0$ for $f(\vec{x})$.
- We choose $\hat{z}$ axis along $\vec{x}$, not the other way around.

$f(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{F}(k) e^{i\vec{k}\cdot\vec{x}} d^3k$

- $\vec{k}$ in spherical coordinate is (k, θ_k, φ_k).
- $\vec{k} = (k_x, k_y, k_z)$ in Cartesian $\Rightarrow$ just a vector!
- $\int_0^\infty k^2 dk \, \frac{1}{k^2+m^2}\left[\frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)\right]$

  odd under $k \leftrightarrow -k$ $\Rightarrow$ both odd $\Rightarrow$ even integrand $\Leftarrow$ formally extend to negative k

- easiest way: change to $k' = -k$ in second integral
- or combine terms to get $\sin kr$, then extend to $-\infty$, then back to $e^{\pm ikr}$
- More checks: proportional to A_0; $f(\vec{x})$ must be purely real [$\tilde{F}(\vec{k})$ is real and $f^*(\vec{x}) = f(\vec{x})$ after $\vec{k} \to -\vec{k}$ in integral]; no dependence on angles of $\vec{k}$ so can't depend on angles on $\vec{x}$ (only on r); inverse transform of answer, $m \to \infty \Rightarrow \tilde{F}(k) \to 0$ so $f(\vec{x}) \to 0$ so e^{+mr}/r must be wrong; $m \to 0$ is well-defined and non-zero, so $m e^{-mr}$ must be wrong (recognize Coulomb FT); units, of course!

- Prob 5: Why no δ-function? $\frac{1-e^{-\alpha r}}{r} \xrightarrow[r\to 0]{} \frac{1-(1-\alpha r)}{r} \to \alpha$ so no $\frac{1}{r}$

  or $\nabla^2 \frac{1}{r} - (\nabla^2 \frac{1}{r}) e^{-\alpha r}$ cancelling δ-functions

  or $\delta^3(\vec{x})(1-e^{-\alpha r}) = 0$

  cancels $\frac{1}{r}$ $\downarrow$ $r^2 dr$

  - Is a charge density $\propto \frac{1}{r}$ unphysical because $\rho \to \infty$ as $r \to 0$? No, because $dQ = \rho \, d^3r$

- Prob 6a: Take R to be entire inner cavity and $\Phi(\vec{x}_0)$ inside (arbitrary)
  - Φ_S is potential on boundary, inner surface of conductor, so constant
  - Earnshaw: $\Phi(\vec{x}_0) \le \Phi_S$, because minimum on boundary, by $\Phi(\vec{x}_0) \ge \Phi_S$ also

  $\Rightarrow \Phi(\vec{x}_0) = \Phi_S$ for any $\vec{x}_0$ in R $\Rightarrow$ $\vec{E} = -\vec{\nabla}\Phi(\vec{x}) = 0$ in R.

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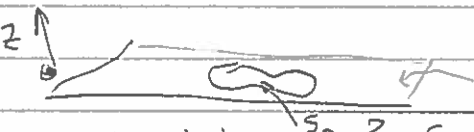
Comments on PS#9 problems: Solving $\nabla^2\Phi=0$ or $\nabla^2\Phi=-\rho/\varepsilon_0$

- In general, can find $\Phi(\vec{x})$ directly or $G_D(\vec{x},\vec{x}')$ and then $\Phi(\vec{x})$ from master formula
  - recall physical meaning of $G_D(\vec{x},\vec{x}')$
    $\Rightarrow$ scalar potential at $\vec{x}$ from unit charge (without $4\pi\varepsilon_0$) at $\vec{x}'$, with $\Phi(\vec{x}_s)\equiv 0$ for $\vec{x}_s$ on S. $\left[\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|}+F(\vec{x},\vec{x}')\right]$
  - several ways to find Φ or G_D:
    - method of images (relevant for spheres in 1 and plane in 2; also 6.)
    - expansion in eigenfunctions (could use for 5,7)
    - eigenfunctions plus division of region (for 3,4)

  $\Rightarrow$ if it works, it's correct $\Rightarrow$ but not always optimal (e.g., may converge slowly)



- Problem 2 — for master formula, S is entire plane
  - do $z>0 \in V$. What is $\hat{n}'$? So what does master formula say?
  - then do $z<0$ (shouldn't need to repeat calculations)
  - Use G_D, because $\Phi(\vec{x}_s)$ is specified.
  - What is G_D? (Remember, even though $\phi_0\neq 0$, $G_D(\vec{x},\vec{x}_s)\equiv 0$; but $\frac{\partial G_D}{\partial n'}\neq 0$!)
  - evaluate $\frac{\partial G_D}{\partial n'}$ in Cartesian coordinates.
- Problem 1

  Just like done earlier in class, except 1(d), which requires thought!

  If sphere is grounded, then where is charge on conductor and how much?

  Test: is there work done moving charge from ∞ to surface? No! $\rightarrow$ Is there an electric field outside? What if not grounded?

  - If sphere is at V and no charge inside, what is $\Phi(\vec{x})$ inside? Now use superposition.
  - If we add Q to the sphere (and keep it isolated), where is the charge on the conductor surfaces?
  - Do either of these changes affect the image charge? Do either lead to a force on q?

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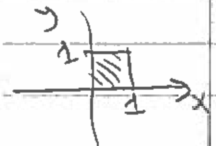

Problem 3 is a 2D version of our rectangular box with $a=b=1$. [We could also have z; just consider independent of z]

· Recall first the expansion in eigenfunctions. We found

vanishes on all sides of box →
$$G_D(\vec{x},\vec{x}') = \frac{32\pi}{abc}\sum_{l,m,n=1}^{\infty}\frac{1}{\pi^2(l^2/a^2+m^2/b^2+n^2/c^2)}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)$$
$$\times \sin\left(\frac{n\pi z}{c}\right)\sin\left(\frac{n\pi z'}{c}\right)$$

Check: $\nabla^2 G_D = \underbrace{-4\pi}_{\text{(from 2nd derivative)}}\Big(\underbrace{\sum_{l=1}^{\infty}\frac{2}{a}\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)}_{\delta(x-x')}\Big)\Big(\underbrace{\qquad}_{\delta(y-y')}\Big)\Big(\underbrace{\qquad}_{\delta(z-z')}\Big)$

· 2D version $G_D(x,y;x',y') = 16\pi\sum_{l,m=1}^{\infty}\frac{1}{\pi^2(l^2+m^2)}\sin(l\pi x)\sin(l\pi x')\sin(m\pi y)\sin(m\pi y')$
$a=b=1$

eigenvalues   eigenfunctions

· Generic expansion for $\hat{\mathcal{L}}_x G_D(x,x') + \lambda w(x) G_D(x,x') = -4\pi\delta(x-x')$

$\frac{d}{dx}f(x)\frac{d}{dx} - g(x)$ set $=1$ for now

Find eigenfunctions of $\hat{\mathcal{L}}_x$:
$$\mathcal{L}_x \varphi_n(x) = \lambda_n \varphi_n(x) \qquad \text{[here } \mathcal{L}_x \to \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \text{ and}$$
and $\qquad\qquad\qquad\qquad\qquad\qquad \varphi_n(x) \to X(x)Y(y) = \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}$]
$$\sum_n \varphi_n(x')\varphi_n(x) = \delta(x-x') \qquad \text{with given BC's}$$

$$\Rightarrow (\hat{\mathcal{L}}_x+\lambda)G_D = -4\pi\mathbb{1} \Rightarrow G_D = (\hat{\mathcal{L}}_x+\lambda)^{-1}(-4\pi\mathbb{1})$$
use $\sum_n |\varphi_n\rangle\langle\varphi_n| = \mathbb{1}$

well-defined in eigenbasis

$$G_D(x,x') = -4\pi\sum_n \frac{\varphi_n(x')\varphi_n(x)}{\lambda+\lambda_n} \quad \text{satisfies all conditions for } G_D!$$

So can apply with $\lambda=0$, $\lambda_n \to \lambda_{lm} = \pi^2(l^2+m^2) \Leftarrow$ eigenvalues of $X(x)Y(y)$ for square
$\mathcal{L}_x = \nabla^2$ [or $\lambda_{lmn} = \pi^2(l^2/a^2+m^2/b^2+n^2/c^2)$ for rectangular box]

Ok, now do the other method → back to (196)

edges of square

· Problem 4. Charge density is $\rho(x,y)=1$, "surfaces" □ are at $\Phi(x_s,y_s)=0$

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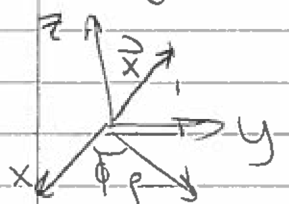
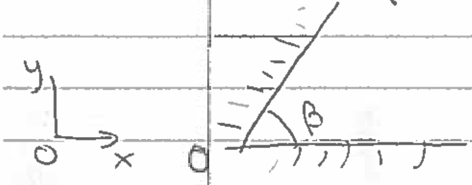
Now let's consider similar developments, but in cylindrical coordinates (separation of variables, expansion in eigenfunctions, division of region)



$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z.$

First consider (following Jackson), a prototype problem independent of z: conducting planes perpendicular to x,y plane at angle β and potential V.



- Why is this good for cylindrical? Really polar coordinates in x-y plane.
- Goal: Solve $\nabla^2\Phi = 0$ where $\Phi = \Phi(\rho,\varphi)$ ← z-independent

Check Jackson covers: $\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$

- Usual game: try separation of variables $\Phi(\rho,\varphi) = R(\rho)\Psi(\varphi)$

$$\Rightarrow \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho R(\rho)\right)\Psi(\varphi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Psi}{\partial\Phi^2} = 0 \Rightarrow \underbrace{\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)}_{=\nu^2} + \underbrace{\frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2}}_{=-\nu^2} = 0$$

separation constants

- Here we anticipate the sign of the constants, but we can always change later if we guess wrong.

- Two equations. Angular one is $\frac{d^2\Psi}{d\varphi^2} + \nu^2\Psi = 0 \Rightarrow \Psi(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi)$ (if $\nu \neq 0$)

Why not $-\nu^2$ and $\Psi(\varphi) \propto e^{\pm\nu\varphi}$? (Come back to this later)

If $\nu = 0$, then $\Psi(\varphi) = A_0 + B_0\varphi$.

- Radial equation: $\underbrace{\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)}\ - \nu^2 R = \rho^2 R'' + \rho R' - \nu^2 R = 0$

  - Recall Sturm-Liouville $\left[\left(\frac{d}{dx}f(x)\frac{d}{dx} - g(x)\right) + \lambda w(x)\right]\Psi(x)$ ← also come back to this

    What are f, g, w? and λ? [It's not so immediate!]

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Can we find $R(\rho)$? If we guess $R(\rho) \propto \rho^{\lambda}$ (based on Frobenius ideas $\Rightarrow \rho^{\lambda} \sum a_n \rho^n$ yields a_0 only)

$$\Rightarrow \lambda(\lambda-1) + \lambda - \nu^2 = 0 \Rightarrow \lambda = \pm\nu \text{ or two solutions for } \nu \neq 0$$

$$\Rightarrow \nu \neq 0 \begin{cases} R(\rho) = a_\nu \rho^{\nu} + b_\nu \rho^{-\nu} \\ \Psi(\varphi) = A_\nu \cos(\nu\varphi) + B_\nu \sin(\nu\varphi) \end{cases}$$

and

$$\nu = 0 \begin{cases} R(\rho) = a_0 + b_0 \ln\rho \\ \Psi(\varphi) = A_0 + B_0\varphi \end{cases} \leftarrow \left[\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = 0 \Rightarrow \frac{dR}{d\rho} = \frac{\text{const}}{\rho} \Rightarrow R = \ln\rho + \text{const}\right]$$

• This is the general form of solutions to problems with z independence.
$\Rightarrow$ write as sum over all allowed values, fix constants by conditions on the problem.

Now apply to  problem $\Rightarrow$ determine constants and ν by BC's:

i) We need $\Phi(\varphi=0) = \Phi(\varphi=\beta) = V$ for all ρ

• To get a constant independent of ρ we need the $\nu=0$ term.

$R(\rho)\Psi(0) = V = (a_0 + b_0\ln\rho)A_0 \Rightarrow a_0A_0 = V,\ b_0 = 0$

$R(\rho)\Psi(\beta) = a_0A_0 = V$ ✓

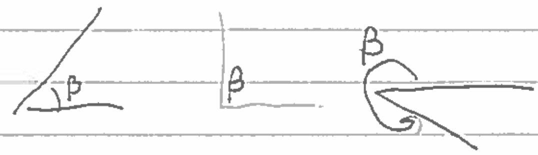
• So all ν terms must yield $\Phi(\varphi=0) = \Phi(\varphi=\beta) = 0$.

$\Rightarrow A_\nu \equiv 0$ to get $\sin(\nu\varphi)$ and $\sin(\nu\beta) = 0 \Rightarrow \nu = \frac{m\pi}{\beta},\ m = 1, 2, \ldots$

full expansion $\Rightarrow$ $$\boxed{\Phi(\rho,\varphi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\varphi\right)}$$

• Don't have $\rho^{-\nu}$, because nonsingular at $\rho = 0$.

• To find the a_m's we would have to look in detail at the large ρ behavior but let's focus on small ρ, where we can ask about the charge density and electric fields for various β's:



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If ρ is small, unless BC's make a_1 zero, the $m=1$ term will dominate (for small enough ρ)

$$\Rightarrow \boxed{\Phi(\rho,\varphi) \doteq V + a_1 \rho^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\varphi\right)}$$

What is the radial $\vec{E}$ field E_ρ?

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\frac{\pi a_1}{\beta} \rho^{\left(\frac{\pi}{\beta}-1\right)} \sin\left(\frac{\pi}{\beta}\varphi\right)$$

same ρ dependence

while the tangential field is

$$E_\varphi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi} = -\frac{\pi a_1}{\beta} \rho^{\left(\frac{\pi}{\beta}-1\right)} \cos\left(\frac{\pi}{\beta}\varphi\right)$$

So if $\beta > \pi$  , the electric field tends to blow up as $\rho \to 0$

Also the surface charge densities at $\phi=0$ and β are equal and approximately

$$\sigma(\rho) = \varepsilon_0 E_\phi(\rho,0) \doteq -\frac{\varepsilon_0 \pi a_1}{\beta} \rho^{(\pi/\beta - 1)}$$

so again we have singular behavior near the corner.

$\Rightarrow$ sharp points lead to large fields and surface charge densities, which in air causes electrical breakdown $\Rightarrow$ lightening rod provides conducting path to ground.

Next time: Z-dependent case!