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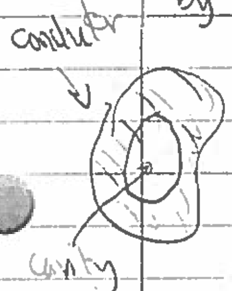
7701 Lecture 30

· Problem set #9 follow-up ...

· In problem 5 we had a cube with four faces grounded and two at potential V. We found numerically that the scalar potential at the center: $\vec{x} = (a/2, a/2, a/2)$ was V/3, which was the average of the potential on each of the conducting sides.

Part c) asked you (for bonus) to prove this. Let's do it by first recalling the Earnshaw theorem problem on the midterm.



· A good, clean answer to this problem:
  · Take region R to be the entire cavity and $\Phi(\vec{x}_0)$ to be the potential at $\vec{x}_0 \in R$.
  · Let V be the potential on the perfect conductor $\Rightarrow$ constant ← (many omit this step)
  · Earnshaw: $\Phi(\vec{x}_0) \leq V$ because maximum on boundary
    but $\Phi(\vec{x}_s) \geq V$ because minimum on boundary
  $\Rightarrow$ for conductor $V \leq \Phi(\vec{x}_s) \leq V$ or $\Phi(\vec{x}_0) = V$ for all $\vec{x}_0$)
  $\Rightarrow \vec{E} = -\vec{\nabla}\Phi = 0$ in R.

· Now apply this to the cube with all sides at same potential V.
  · What is $\Phi(\vec{x}_{center})$? Satisfies conditions $\Rightarrow$ V
  · But what if V on one side only? By symmetry, same for any side but by superposition adding each in turn must give V again $\Rightarrow$ for one side V/6.
  · Now by superposition $\Phi_{center} = \sum_{i=1}^{6} \frac{1}{6} V_i = \frac{1}{6}\sum_{i=1}^{6} V_i = \langle V_i \rangle$ ← average of faces
  · The generalization to any regular polyhedron is immediate!

· Now look at ps9_checks.nb notebook. Two ways to find $\Phi(\vec{x})$ in the cube from the master formula: $G_D \sim \sum_{k,l,m} \sin^6$ or $G_D \sim \sum_{k,l} \sin^4 \cdot \sinh^2$
  · both give same answer (anywhere in cube) if enough terms included.
  · but $\sin^6$ requires much more than $\sin^4 \cdot \sinh^2$! So it matters a lot in practice, but why?

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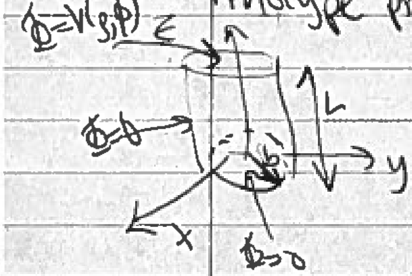
Now let's think about problems that are natural in cylindrical coordinates but which have z dependence.

Prototype problem (analogous to rectangular box for Cartesian coordinates):



A cylinder of finite height L and radius b, on which we specify the potential on the cylinder surface and ends.

- e.g. grounded except for $V(\rho,\phi)$ on the top.

- If we have separated the ρ, ϕ, and z variables, we write a general expansion from solutions to each of the equations and determine the unknown coefficients and separation constants using the boundary conditions.
  - Same as before, only we need to identify the equations and their solutions.
  - As for the box, we'll be able to do a direct expansion or solve via Dirichlet Green functions.

- Ok, so we need the full Laplacian now!

$$\nabla^2\Phi = \frac{\partial^2}{\partial\rho^2}\Phi + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\Phi + \frac{\partial^2}{\partial z^2}\Phi = 0 \qquad \Phi = \Phi(\rho,\varphi,z)$$

Separation ansatz: $\Phi(\rho,\varphi,z) = R(\rho)Q(\varphi)Z(z)$

divide by RQZ

$$\Rightarrow \frac{R''QZ}{RQZ} + \frac{\frac{1}{\rho}R'QZ}{RQZ} + \frac{\frac{1}{\rho^2}RQ''Z}{RQZ} + \underbrace{\frac{RQZ''(z)}{RQZ(z)}}_{\text{constant because only } z \text{ dependence}} = 0$$

First: $\frac{d^2Z}{dz^2} = k^2 Z$

$$\Rightarrow Z(z) = C\sinh(kz) + D\cosh(kz) \quad \text{or} \quad e^{\pm kz}$$

- for the prototype problem, $D = 0$ since $Z(z=0) = 0$.

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Now we have $\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} + \chi^2 = 0$

$\Rightarrow$ multiply by ρ^2 to isolate φ and ρ dependence

$$\underbrace{\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + \chi^2\rho^2}_{+\nu^2} + \underbrace{\frac{1}{\rho^2}\frac{Q''}{Q}}_{-\nu^2} = 0$$

$\leftarrow$ separation constants

$\Rightarrow \frac{d^2Q(\varphi)}{d\varphi^2} + \nu^2 Q(\varphi) = 0 \Rightarrow Q(\varphi) = A\sin(\nu\varphi) + B\cos(\nu\varphi)$ or $e^{\pm i\nu\varphi}$

- sines and cosines because we want single-valued functions in φ $\Rightarrow$ also ν will be integers for our prototype problem (and many others!)

- So our only "new" equation, multiplying by R, is:

$$\boxed{\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (\chi^2\rho^2 - \nu^2)R = 0}$$

It looks like both χ and ν are relevant, but each of the first two term is unchanged under $\rho \Rightarrow \alpha\rho$ so lets rescale with $\boxed{x = \chi\rho}$

$$\Rightarrow x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - \nu^2)R = 0$$

or $\boxed{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0}$ (Is This in Sturm-Liouville form?)

- The solutions to This differential equation are Bessel functions, but suppose we don't know that.

$\Rightarrow$ go back to our old friend Frobenius

look for series solution $\Rightarrow R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$ with $a_0 \neq 0$

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Recall that we substitute into the equation

$$\sum_{j=0}^{\infty}\left[a_j(j+\alpha)(j+\alpha-1)x^{\alpha+j-2}+a_j(j+\alpha)x^{\alpha+j-2}-\nu^2 a_j x^{\alpha+j-2}\right]+\sum_{j=0}^{\infty}a_j x^{\alpha+j}=0$$

cancels

$x^{\alpha-2}$: $a_0(\alpha^2-\nu^2)x^{\alpha-2}=0 \Rightarrow \alpha=\pm\nu$ "indicial equation"

$x^{\alpha-1}$: $a_1[(\alpha+1)^2-\nu^2]=0$ plug in $\alpha=\pm\nu \Rightarrow a_1=0$

general: $a_{j+2}[(j+\alpha+2)^2-\nu^2]+a_j=0$

$$\Rightarrow a_{j+2}=-a_j\frac{1}{(j+2)^2+2\alpha(j+2)} \quad \text{or} \quad \boxed{a_j=-\frac{a_{j-2}}{j(j+2\alpha)}}$$

Now for odd $j=2n+1$, $a_{2n+1}=0$, so we can write

$\boxed{a_{2n}=-\frac{a_{2n-2}}{4n(\alpha+n)}}$ for integer n. Given $a_0\to a_2\to a_4\to\ldots$

- We can see that the coefficients are going to have powers of 2 from the 4 and a factorial from $n\cdot(n-1)\cdots$ and then something like a factorial from $(\alpha+n)\cdot(\alpha+n-1)\cdot(\alpha+n-2)\cdots$
- Recall that we can use the gamma function to write these generalized factorials

$$\Gamma(z)\equiv\int_0^{\infty}t^{z-1}e^{-t}\,dt \quad \text{for } z>0$$

and analytically continued to $z\le 0$ using $z\Gamma(z)=\Gamma(1+z)$.

- $\Gamma(m+1)=m!$ for integer m.

- Here we find $a_{2n}=(-1)^n\frac{1}{4^n}\frac{\alpha!}{n!\,(n+\alpha)!}\cdot a_0$ ← choose to be $\frac{1}{2^{\alpha}\alpha!}$ (convention)

$(n+\alpha)! = \Gamma(\alpha+n+1)$, $n! = \Gamma(n+1)$, $\alpha! = \Gamma(\alpha+1)$

$$\Rightarrow \boxed{a_{2n}=(-1)^n\frac{1}{2^{\alpha+2n}\Gamma(n+1)\Gamma(\alpha+n+1)}}$$

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With $\alpha = \pm\nu$ these define the Bessel functions of the 1st kind

uppercase J →

$$\boxed{\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n} \end{aligned}}$$

- Now we're going to need $\nu = m$ for our cylinder (because we need $e^{\pm im\phi}$ for single-valued)
  - But in this case

$$J_{-m}(z) = (-1)^n J_m(z)$$

$\Rightarrow$ we introduce an independent function

Neumann function: $N_\nu(x) = \dfrac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}$ (Bessel function of the 2nd kind)

which is well behaved for $\nu \to$ integer.

- We can derive (but we'll look them up for the most part) various useful properties of Bessel functions.

- Recursion relations

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\frac{d\Omega_\nu(x)}{dx}$$

works for J_ν, N_ν and also

$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$

$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$

"Hankel functions"

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Small and large x limits

$$\begin{cases} J_\nu(x) \xrightarrow[x\ll 1]{} \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu \\ N_\nu(x) \xrightarrow[x\to 0]{} \begin{cases} \dfrac{2}{\pi}\left[\ln\dfrac{x}{2} + 0.577\ldots\right], & \nu=0 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu, & \nu\neq 0 \end{cases} \end{cases}$$

← "regular" at origin for $\nu>0$

$J_m(x) \propto x^m$ at small x $(m\geq 0)$

$$\begin{cases} J_\nu(x) \xrightarrow[x\to\infty]{} \sqrt{\dfrac{2}{\pi x}}\cos\left(x-\dfrac{\nu\pi}{2}-\dfrac{\pi}{4}\right), & x\gg 1 \\ N_\nu(x) \xrightarrow[x\to\infty]{} \sqrt{\dfrac{2}{\pi x}}\sin\left(x-\dfrac{\nu\pi}{2}-\dfrac{\pi}{4}\right), & x\gg 1 \end{cases}$$

} look like damped cosines and sines

* See Mathematica for graphs

- The roots (zeros) of J_ν will turn out to be relevant. We define $x_{\nu n}$ to be the n^{th} root $(n=1,2,3)$: $J_\nu(x_{\nu n})=0$.

- Most important orthogonality relations (proved later)

$$\int_0^a J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\rho\, d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2\delta_{n'n}$$

expand $f(\rho)$ on $0\leq\rho\leq a$ $\Rightarrow$ $\boxed{f(\rho)=\sum_{n=1}^{\infty} A_{\nu n}J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)}$ "Fourier-Bessel series"

where $\boxed{A_{\nu n}=\dfrac{2}{a^2 J_{\nu+1}^2(x_{\nu n})}\displaystyle\int_0^a f(\rho)\,J_\nu\!\left(\frac{x_{\nu n}\rho}{a}\right)\rho\, d\rho}$

- especially useful when $f(a)=0$.
- how do we know this is complete?

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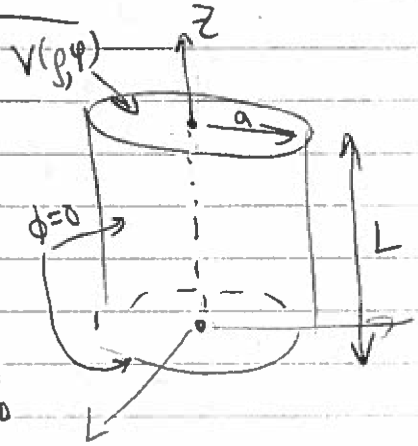
- We also have $\frac{1}{k}\delta(k-k') = \int_0^\infty dx\, x\, J_\nu(kx) J_\nu(k'x)$

- Modified Bessel functions for $\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$ ← opposite k^2

$$\Rightarrow \boxed{\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix) \\ K_\nu(x) &= \frac{\pi}{2} i^{\nu+1}\left[J_\nu(ix) + iN_\nu(ix)\right] \end{aligned}}$$

- Now solve our cylinder



$$Q(\varphi) = A \sin(\overset{\nu=m}{m\varphi}) + B\cos(m\varphi)$$

$$Z(z) = C\sinh(kz) + \underset{\text{from } Z(0)=0}{\overset{0}{\cancel{D}}}\cosh(kz)$$

$$R(\rho) = E J_m(k\rho) + \overset{0 \text{ because regular (finite) at } \rho=0}{\cancel{F}} N_m(k\rho)$$

$$R(a) = 0 \Rightarrow k \to k_{mn} = \frac{x_{mn}}{a} \leftarrow \text{roots}$$

expansion $\Rightarrow \Phi(\rho,\varphi,z) = \sum_{m=0}^\infty \sum_{n=1}^\infty J_m(k_{mn}\rho)\sinh(k_{mn}z)\cdot\left[A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\right]$

BC: $\Phi(\rho,\varphi,z=L) = V(\rho,\varphi)$

$$\Rightarrow V(\rho,\varphi) = \sum_{m,n} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left[A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right]$$

inverting $\Rightarrow \begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{2}{\pi a^2 \sinh(k_{mn}L) J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\varphi \int_0^a d\rho\, \rho\, V(\rho,\varphi) J_m(k_{mn}\rho) \begin{pmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{pmatrix}$

and for $m=0$ use $\frac{1}{2}B_{0n}$.