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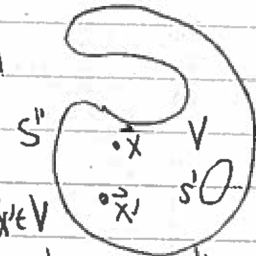
7701 Lecture 33

Plan: Recap and loose ends from previous notes.

PS#10 Problem 6 is a good problem for checking your physical understanding of Green functions and also what happens with conductors.

We are asked to use the physical interpretation of the Dirichlet Green function $G_D(\vec{x},\vec{x}')$ to show that $\Lambda(\vec{x},\vec{x}')<0$, where

$$G_D(\vec{x},\vec{x}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{x}'|} + \Lambda(\vec{x},\vec{x}') \quad \vec{x},\vec{x}'\in V$$



The picture at the right defines a possible V, where we allow for strange shapes and disconnected surfaces (like S').

- The physical interpretation is that $G_D(\vec{x},\vec{x}')$ is the scalar potential at $\vec{x}$ due to a unit point charge at $\vec{x}'$ in the presence of a perfect, grounded conductor at the surface S of the volume V (including S' as part of the surface).
- This conforms to G_D being zero on the surface ("grounded")

So now we ask: what is the induced charge density $\sigma(\vec{x}_s)$ on the surface(s)? Claim: it is negative (or zero) everywhere.

- Why couldn't it be positive in some remote part? Think of field lines for the electric field. They start on positive charges and end on negative charges. If there is a region of positive σ, then the field lines in V from there must also end on the surface. If so, then $\int \vec{E}\cdot d\vec{l}$ along this line is positive definite, but also equal to the potential difference, which is zero (perfect grounded conductors) $\Rightarrow$ contradiction!
  $\Rightarrow$ the surface charge is negative or zero.

$\int \vec{E}\cdot d\vec{l}$ on this line would be non-zero $\Rightarrow \Delta V \neq 0$

- But if the charge is negative, the scalar potential is also negative everywhere (e.g. from the integral for the potential).

QED

For part b) see the solutions to PS#10.

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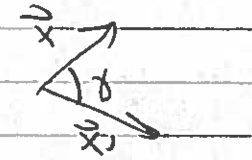
Another follow-up to PS#10, this time to problem 3.

- Here the issue is calculating

$$\int_0^1 P_\ell(x)\,dx$$

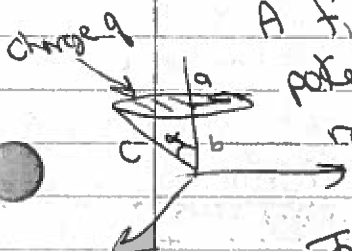
- see 233 for options. One is recursion relations. Where do they come from?
- As a set up, we return to 222 and the expansion for the free Green function:

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$



which we can apply in unexpected ways.

A first example is on 223, which is to find the potential anywhere due to a disk of charge q and radius "a" at height b above the origin.



- We know that we have the expansion (in spherical coords):

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

but how do we find the coefficients?

- The first key is to use the symmetry to solve a special case problem: here r on the z-axis $\Rightarrow \theta = 0$

$$\Rightarrow \Phi(r,0) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+c^2-2rc\cos\alpha}}$$ (e.g. from the basic integration)

- The second key is to recognize this is the same as a free Green function $\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$

with c taking the place of r' and $\gamma \to \alpha$.

$$\Rightarrow \Phi(r,0) = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha)$$ where $r_< = \min\{r,c\}$ and $r_> = \max\{r,c\}$

So finally

$$\boxed{\Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha)\, P_\ell \cos\theta}$$

which works everywhere (whether $r<c$ or $r>c$!).

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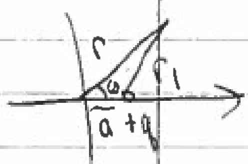
Now for recursion relations, we use a generating function for Legendre polynomials, as detailed on 228 - 229.



- The key to establishing the generating function is to recognize that another basic problem: the potential due to a point charge away from the origin (figure at left) can use the expansion for $1/|\vec{x}-\vec{x}'|$ as well.
- Note the results we can get:
  - $P_\ell(1)=1$, $P_\ell(-1)=(-1)^\ell$ for all ℓ at the same time!
  - recursion relations with and without derivatives.
  - $P_\ell(-x)=(-1)^\ell P_\ell(x)$
  - $|P_\ell(\cos\theta)| \le P_\ell(1) = 1$

see list on 229

- Next let's extend the expansion in Legendre polynomials, which applies for $\Phi(\vec{x})$ when we have azimuthal symmetry (so the expansion is in $P_\ell(\cos\theta)$) or for $1/|\vec{x}-\vec{x}'|$, where the expansion is in $P_\ell(\cos\gamma)$ where γ is the angle between $\vec{x}$ and $\vec{x}'$.
  - We would like to use θ,φ of $\vec{x}$ and θ',φ' of $\vec{x}'$
    $\Rightarrow$ use the addition theorem:
    $$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi')Y_{\ell m}(\theta,\varphi)$$
  - note that $P_\ell(\cos\gamma)$ real $\Rightarrow$ we can put the $*$ on either $Y_{\ell m}$.
    $$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\varphi)Y^*_{\ell m}(\theta',\varphi')$$
    [see 237 for other expansions]
  - The $Y_{\ell m}$'s (or Y_ℓ^m's), called "spherical harmonics", arise when we do the full separation of variables in spherical coordinates.
    $$\nabla^2\Phi(r,\theta,\varphi) = \frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial\Phi}{\partial r}\right) + \underbrace{\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)}_{\equiv -\frac{L^2}{\hbar^2}}\Phi$$

where $\vec{L} \equiv \frac{\hbar}{i}\vec{r}\times\vec{\nabla}$ from quantum mechanics, which implies

$\frac{1}{\hbar}L_z = -i\frac{\partial}{\partial\varphi} \Rightarrow \frac{1}{\hbar}L_z Y_{\ell m} = mY_{\ell m}$ and $\frac{1}{\hbar^2}L^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}$

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So the Y_{lm}'s are eigenstates of both L^2 and L_z.
They form a complete set for expanding any function of θ and φ.
- The $l=0, m=0$ $Y_{00} = \frac{1}{\sqrt{4\pi}}$ is independent of angle.
It is normalized. In general,

$$\boxed{\int d\Omega\, Y^*_{lm}(\theta,\varphi) Y_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}}$$

$\int d\Omega = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi$

- The $l=1$ Y_{lm}'s times r given z, x, and y:

$$rY_{10} = \sqrt{\frac{3}{4\pi}}\, r\cos\theta = \sqrt{\frac{3}{4\pi}}\, z$$

$$rY_{1+1} = -\sqrt{\frac{3}{8\pi}}\, r\sin\theta\, e^{i\varphi} \qquad rY_{1-1} = -\sqrt{\frac{3}{8\pi}}\, r\sin\theta\, e^{-i\varphi}$$

$$\Rightarrow r(Y_{1+1} \pm Y_{1-1}) \propto \begin{Bmatrix} x \\ y \end{Bmatrix}$$

- An application of the expansion with spherical harmonics for $\Phi(\vec{x})$ is given on 234 - 236. The general expansion is

$$\boxed{\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) Y_{lm}(\theta,\varphi)}$$

where A_{lm} and B_{lm} are determined by BC's (e.g. regular at origin or going to zero at ∞) along with projection:

$$\boxed{\int d\Omega\, Y^*_{l'm'}(\theta,\varphi)\, \Phi(r,\theta,\varphi) = A_{l'm'} r^{l'} + B_{l'm'} r^{-(l'+1)}}$$