Physics 7701: Problem Set #1

The problems are due in Prof. Furnstahl's mailbox in the main office by 4pm on Tuesday, August 27. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*30 pts*) **Practice with δ_{ij} and ϵ_{ijk}.** Do these problems using δ's and ϵ's.

   (a) Show that

   $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \qquad (1)$$

   (b) Prove the identity

   $$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}) \qquad (2)$$

   (Hint: Start with the last two terms on the right-hand side.)

   (c) A distribution of electric currents creates a constant magnetic momentum $\mathbf{m} = \text{const}$. The force on $\mathbf{m}$ in an external magnetic induction $\mathbf{B}$ is given by

   $$\mathbf{F} = \boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{m}) \ . \qquad (3)$$

   Show that (given $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$)

   $$\mathbf{F} = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B} \ . \qquad (4)$$

2. (*10 pts*) **Functions in the complex z plane.**

   (a) Find *all* solutions to $z^5 = -1$ and plot in complex plane.

   (b) Find *all* solutions to $\cos z = 100$.

3. (*10 pts*) Small amplitude waves in a plasma are described by the relations

   $$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n_0 v) &= 0 &(5)\\ \epsilon_0\frac{\partial E}{\partial x} &= -en &(6)\\ m\frac{\partial v}{\partial t} &= -eE - m\nu v \ , &(7) \end{aligned}$$

   where n_0, e, m, ν, and ϵ_0 are constants. The constant ν is the collision frequency. Assume that n, E, and v are all proportional to $\exp(ikx - i\omega t)$.

(a) Solve the equations for nonzero n, E, and v to show that ω satisfies the equation

$$\omega^2 + i\nu\omega = \frac{n_0 e^2}{m\epsilon_0} \equiv \omega_p^2 \, , \tag{8}$$

where ω_p is the plasma frequency.

(b) Solve this equation to find the frequency ω and explain how this shows that collisions damp the waves (assume $\nu \ll \omega_p$).

4. (*20 pts*) **Cauchy-Riemann relations.**

   (a) Find the analytic function $w(z) = u(x, y) + iv(x, y)$ if $v(x, y) = e^{-y} \sin x$.

   (b) One of the functions $u_1 = 2(x - y)^2$ and $u_2 = \frac{x^3}{3} - xy^2$ is the real part of an analytic function $w(z) = u + iv$. Which is it? Find the function $v(x, y)$ and write w as a function of z.

5. (*10 pts*) Determine the Taylor or Laurent series for each of the following functions in the immediate neighborhood of the point specified. This means to find the general term (as a function of an index n) and not just the first few terms. In each case, determine the region of convergence of the series. Can you check your answers with Mathematica?

   (a) $\dfrac{\cos z}{z - 1}$ about $z = 1$

   (b) $\dfrac{\ln z}{z - 1}$ about $z = 1$

Optional problems (count as bonus points)

6. (*5 pts*) Show that $|\sin z| \geq |\sin x|$ for all z, where x is the real part of z.

7. (*10 pts*) For each of the following complex functions, find the real and imaginary parts u and v (i.e., $f = u + iv$) and show that u and v obey the Cauchy-Riemann relations. Then find the derivative df/dz directly by differentiating with respect to z and *also* by differentiating with respect to x and y and expressing the answer in terms of z. Do they agree?

   (a) $f = z^2 \sin z$

   (b) $f = \dfrac{1}{1 + z}$

8. (*10 pts*) Investigate the function $w = 1/\sqrt{z}$.

   (a) Find the real-valued functions $u(r, \theta)$ and $v(r, \theta)$, where $w = u + iv$.

   (b) How many branches does this function have?

   (c) Find the image of the unit circle under this mapping.