

Physics 7701: Problem Set #10

These problems are due in Russell Colburn's mailbox in the main office by 4:30pm on Thursday, November 21. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (20 pts) **Potential in cylinder** (Jackson 3.9). A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential valid anywhere inside the cylinder.
2. (20 pts) **Cylinder again** (Jackson 3.10). For the cylinder in the last problem, the cylindrical surface is made of two equal half cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & \text{for } -\pi/2 < \phi < \pi/2 \\ -V & \text{for } \pi/2 < \phi < 3\pi/2 \end{cases} . \quad (1)$$

- (a) Find the potential inside the cylinder.
- (b) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with the two-dimensional result (from Jackson 2.13)

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1} \left(\frac{2b\rho}{b^2 - \rho^2} \cos \phi \right) \quad (2)$$

3. (20 pts) **Concentric spheres** (Jackson 3.1). Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V . The other hemispheres are at zero potential.
 - (a) Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$.
 - (b) Check your solution against known results in the limiting cases $b \rightarrow \infty$, and $a \rightarrow 0$.
 - (c) (Bonus) Carry out additional checks of your answer.
4. (20 pts) **Spherical surfaces** (Jackson 3.2). A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} [P_{l+1}(\cos \alpha) - P_{l-1}(\cos \alpha)] \frac{r^l}{R^{l+1}} P_l(\cos \theta) \quad (3)$$

where, for $l = 0$, $P_{l-1}(\cos \alpha) = -1$.

- (b) What is the potential outside?
 (c) Find the magnitude and direction of the electric field at the origin.
 (d) Discuss the limiting forms of the potential (parts a and b) and the electric field (part c) as the spherical cap becomes (i) very small, and (ii) so large that the area with charge on it becomes a very small cap at the south pole.

Optional problems (count as bonus points)

5. (10 pts) **Free-space Green function in polar coordinates** (Zangwill 8.18). The free-space Green function in two dimensions (potential of a line charge) is (Zangwill conventions!)

$$G_0^{(2)}(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi\epsilon_0} \ln |\mathbf{x} - \mathbf{x}'|. \quad (4)$$

Use the division-of-region (direct integration) method to reduce the two-dimensional equation

$$\epsilon_0 \nabla^2 G_0^{(2)}(\mathbf{x}, \mathbf{x}') = -\delta^2(\mathbf{x} - \mathbf{x}') \quad (5)$$

to a one-dimensional equation and establish the alternative representation

$$G_0^{(2)}(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi\epsilon_0} \ln \rho_{>} + \frac{1}{2\pi\epsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\rho_{<}^m}{\rho_{>}^m} \cos m(\phi - \phi'). \quad (6)$$

6. (10 pts) **Green function inequalities** (Zangwill 8.14). The Dirichlet Green function for any finite volume V can always be written in the form (Zangwill conventions)

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|} + \Lambda(\mathbf{x}, \mathbf{x}') \quad \text{where } \mathbf{x}, \mathbf{x}' \in V. \quad (7)$$

The function $\Lambda(\mathbf{x}, \mathbf{x}')$ satisfies $\nabla^2 \Lambda(\mathbf{x}, \mathbf{x}') = 0$.

- (a) Use the physical meaning of the Dirichlet Green function to prove that

$$G_D(\mathbf{x}, \mathbf{x}') < \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \quad (8)$$

- (b) Use Earnshaw's theorem to prove

$$G_D(\mathbf{x}, \mathbf{x}') > 0. \quad (9)$$