Physics 7701: Problem Set #11

These problems are due in Russell Colburn's mailbox in the main office by 10:00am on Tuesday, December 3. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*20 pts*) **Hollow sphere** (Jackson 3.5). A hollow sphere of inner radius a has the potential specified on its surface to be $\Phi = V(\theta, \phi)$. Prove the equivalence of the two forms of solution for the potential inside the sphere:

   (a)
   $$\Phi(\mathbf{x}) = \frac{a(a^2 - r^2)}{4\pi} \int \frac{V(\theta', \phi')}{(r^2 + a^2 - 2ar\cos\gamma)^{3/2}} d\Omega' \tag{1}$$
   where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.

   (b)
   $$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi) \tag{2}$$
   where $A_{lm} = \int d\Omega' \, Y^*_{lm}(\theta', \phi') V(\theta', \phi')$.

2. (*20 pts*) **Point charge potentials** (Jackson 3.6). Two point charges q and $-q$ are located on the z axis at $z = +a$ and $z = -a$, respectively.

   (a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

   (b) Keeping the product $qa \equiv p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the z axis and its potential.

   (c) Suppose now that the dipole of part (b) is surrounded by a *grounded* spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.

3. (*20 pts*) **Multipole theorem** (Jackson 4.4a). Prove the following theorem: For an arbitrary charge distribution $\rho(\mathbf{x})$ the values of the $(2l+1)$ moments of the first nonvanishing multipole are independent of the origin of the coordinate axes, but the values of all higher multipole moments do in general depend on the choice of origin. (The different moments q_{lm} for fixed l depend, of course, on the orientation of the axes.)

4. (*20 pts*) **Multipole expansion example** (Jackson 4.4a,b). A localized distribution of charge has a charge density

$$\rho(\mathbf{x}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta \,. \tag{3}$$

This is the charge density for the $m = \pm 1$ states of the $2p$ level in hydrogen, where the unit of charge is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529 \times 10^{-10}$ m.

(a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\mathbf{x}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right] \,. \tag{4}$$

Optional problems (count as bonus points)

5. (*20 pts*) We may model the force between particles in an atomic nucleus with a three-dimensional square-well potential

$$V(r) = \begin{cases} -V_0 & \text{for } 0 \le r < a \\ 0 & \text{for } r > a \end{cases} \tag{5}$$

Schrödinger's equation for this system takes the form

$$\left(\nabla^2 - 2\frac{m}{\hbar^2}V(r)\right)\psi(\mathbf{x}) = -2\frac{m}{\hbar^2}E\psi(\mathbf{x}) \,, \tag{6}$$

where the particle's energy E is negative.

(a) Write the differential operator in spherical coordinates and show that the general solutions inside and outside the square well may be written in terms of spherical Bessel functions and spherical harmonics. We require $\psi(0)$ be finite and $\psi(\mathbf{x}) \to 0$ as $x \to \infty$. You don't need to re-derive the radial part of the Laplacian (although it won't hurt :), but show how it can be transformed with the given potential to be the equation for spherical Bessel functions.

(b) With $\alpha^2 \equiv 2(m/\hbar^2)V_0 a^2$ and $\epsilon = -E/V_0$ with $E < 0$, show that the energy levels are determined by the equation

$$\sqrt{1-\epsilon}\, k_l(\alpha\sqrt{\epsilon})\, j_{l+1}(\alpha\sqrt{1-\epsilon}) = \sqrt{\epsilon} j_l(\alpha\sqrt{1-\epsilon}) k_{l+1}(\alpha\sqrt{\epsilon}) \,, \tag{7}$$

where k_l is a modified spherical Bessel function (check Arfken for formulas). The boundary condition at $r = a$ is that the wave function and its radial derivative be continuous there.

(c) Use Mathematica to find the energy of the lowest energy level for $l = 0$, $\alpha^2 = 10$. Attach a printout.

6. (*20 pts*) Find the Dirichlet Green's function for Poisson's equation in the interior of a hemisphere of radius a.

   (a) First define the hemisphere by $0 \le \phi \le \pi$ and $0 \le \theta \le \pi$.

   (b) Now repeat the derivation but with the hemisphere defined by $0 \le \theta \le \pi/2$ and $0 \le \phi \le 2\pi$.

   (c) Use one of these two Green's functions (your choice!) to evaluate the potential inside the hemisphere if $\Phi = 0$ on the spherical surface and $\Phi(r) = V_0(1 - r/a)$ on the flat face.