Physics 7701: Problem Set #2

The problems are due in Russell Colburn's mailbox in the main office (or in his office M2060) by 4pm on Wednesday, September 4 (because of the holiday). Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*10 pts*) Find the residue of each of the following functions at the point specified.

   (a) $\dfrac{z-2}{z^2-1}$ at $z=1$

   (b) $\dfrac{\cos z}{1/2-\sin z}$ at $z=\pi/6$

2. (*10 pts*) First pass: Evaluate the following integrals.

   (a) $\oint_C \dfrac{\cos z}{z}\,dz$ where C is a circle of radius 2 centered at the origin.

   (b) $\int_{-\infty}^{+\infty} \dfrac{1}{x^2+2}\,dx$ by the residue theorem (just like in class!).

3. (*30 pts*) Second pass: Evaluate the following integrals using contour integration. Be sure to include all parts of the contour and if you set some part(s) to zero, give a justification. *Verify each of your answers with Mathematica, giving the command you used (preferably on a separate printed sheet but writing it by hand is acceptable).*

   (a) $\int_0^{+\infty} \dfrac{x^2}{1+x^4}\,dx$

   (b) $\int_0^{+\infty} \dfrac{x^{1/3}}{x^2+1}\,dx$ (do this without changing variables such as $x=y^3$)

   (c) $\int_0^{\pi} \dfrac{1}{1+\cos^2\theta}\,d\theta$

4. (*10 pts*) The unit step (θ) function is defined for real a as

$$\theta(t-a) = \begin{cases} 0\,, & t<a\,; \\ 1\,, & t>a\,. \end{cases} \tag{1}$$

   Show that $\theta(t)$ has the integral representation

$$\theta(t) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{ikt}}{k-i\epsilon}\,dk \tag{2}$$

5. (*10 pts*) Show that

$$\int_0^{\infty} \sin(x^2)\,dx = \int_0^{\infty} \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{3}$$

   using the contour in Arfken Fig. 7.15 (6th Ed.) [or Fig. 11.30 (7th Ed.)]. (This result has numerous applications in physics—for example, in signal propagation.)

Optional problems (count as bonus points)

6. (*5 pts*) Find the residue of $\frac{\sin z}{z^2}$ at the origin

7. (*5 pts*) Use the calculus of residues to prove the identity:

$$\int_0^{\pi} d\theta \, \cos^{2n}(\theta) = \pi \frac{(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n-1)!!}{(2n)!!} \quad n = 0, 1, 2, \cdots \tag{4}$$

   (Note: the double factorial is defined in Arfken 8.1.)

8. (*5 pts*) In the quantum theory of atomic collisions we encounter the integral:

$$I = \int_{-\infty}^{+\infty} \frac{\sin t}{t} e^{ipt} \, dt \tag{5}$$

   with p real. Show that:

$$I = \begin{cases} 0 \, , & |p| > 1 \, ; \\ \pi \, , & |p| < 1 \, . \end{cases} \tag{6}$$

   What happens if $p = \pm 1$?