Physics 7701: Problem Set #3

The problems are due in Russell Colburn's mailbox in the main office by 4pm on Tuesday, September 10. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*20 pts*) Contour integral follow-ups:

   (a) Find $\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}}\, dx$ where b is real and $0 < \mathrm{Re}(a) < b$. Use a rectangular contour.

   (b) Show that the unit step function $\theta(t)$ has the alternative integral representation

   $$\theta(t) = \frac{1}{2} + \frac{1}{2\pi i}\mathcal{P}\int_{-\infty}^{+\infty} \frac{e^{ikt}}{k}\, dk \; . \tag{1}$$

2. (*10 pts*) Use the series method to find a solution of Laguerre's differential equation

   $$xy'' + (1-x)y' + \alpha y = 0 \tag{2}$$

   that is regular at the origin. Show that if α is an integer k, then this solution is a polynomial of degree k.

3. (*10 pts*) Solve the Bessel equation

   $$4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \tag{3}$$

   as a Frobenius series in powers of x. Sum the series to obtain close-form expressions for the two solutions.

4. (*10 pts*) Determine a solution of the equation

   $$(1+x)y'' + (3+2x)y' + (2+x)y = 0 \tag{4}$$

   at large x. Use this to determine the solutions for positive x. (Hint: you should find a second-order equation with constant coefficients, which you can solve with an e^{sx} ansatz. The remaining differential equation can be solved by direct integration.)

5. (*10 pts*) Weber's equation is

   $$y'' + \left(n + \frac{1}{2} - \frac{x^2}{4}\right) y = 0 \; , \tag{5}$$

   where n is an integer. Show that the substitution $y = \exp(-ax^2)v(x)$ simplifies this equation for a choice of a that you are to determine. Find two solutions for $v(x)$ as power series in x.

Optional problem (counts as bonus points)

6. (*10 pts*) **Langmuir waves.** Waves in a plasma may be described by a wave form $n = n_0 e^{ikx - i\omega t}$, where the relation between ω and k (which is called the *dispersion relation*) is given by

$$0 = 1 + \frac{\omega_p^2}{k} \int_{-\infty}^{+\infty} \frac{\partial f(v)/\partial v}{\omega - kv}\, dv \tag{6}$$

where ω_p is the plasma frequency $ne^2/\varepsilon_0 m$ and $f(v)$ is the one-dimensional Maxwellian

$$f(v) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{mv^2}{2k_B T}\right) . \tag{7}$$

Notice that the integrand has a singularity at $v = \omega/k$, which is on the real axis, if ω and k are both real. Landau showed that the integral is to be regarded as an integral along the real axis in the complex v-plane, and that the correct integration path passes around and *under* the pole.

(a) Show that the integral may be expressed as

$$\int_{-\infty}^{+\infty} \frac{\partial f(v)/\partial v}{\omega - kv}\, dv = P \int_{-\infty}^{+\infty} \frac{\partial f(v)/\partial v}{\omega - kv}\, dv - \frac{i\pi}{k} \left.\frac{\partial f}{\partial v}\right|_{v=\omega/k} \tag{8}$$

(b) Evaluate the principal value approximately, assuming $\omega/k \gg v_T = \sqrt{k_B T/m}$. *Hint:* First integrate by parts, and then expand the denominator in a series in powers of kv/ω. Neglect the small effect of the pole, and find the frequency ω as a function of k. Now include the imaginary part due to the pole at ω/k. Show that the wave is damped. The result has been confirmed experimentally.

(c) How would the result change if the path of integration were to pass over, rather than under, the pole?

[Note: You may find Section 2.1.4 of Lea Chapter 2 to be useful.]