Physics 7701: Problem Set #4

The problems are due in Russell Colburn's mailbox in the main office by 4pm on Wednesday, September 18. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*10 pts*) The Schrödinger equation in one dimension has the form

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E - V)\psi = 0 \; . \tag{1}$$

   Develop a series solution for ψ in the case where V is the potential due to the interaction of two nucleons:

$$V(x) = C\frac{e^{-\alpha x}}{x} \; . \tag{2}$$

   Assume $\psi(0) = 0$. Obtain at least the first three nonzero terms.

2. (*15 pts*) Develop the full Fourier series for the function $f(x) = x$

   (a) over the range $0 \le x \le 1$;

   (b) over the range $-1 \le x \le 1$.

   (c) Use Mathematica to make a plot (attach a printout) showing the original function and the sum of the first three nonzero terms in each series. Comment on the similarities and differences between the two series.

3. (*10 pts*) Find an exponential Fourier series for the function $\sinh \alpha x$ on the range $0 \le x \le 2\pi$. By combining terms, rewrite your answer as a series in sines and cosines.

4. (*10 pts*) A spring-and-dashpot system satisfies the equation

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + k^2 x = f(t) \; . \tag{3}$$

   The system is driven by a periodic driving force with period T:

$$f(t) = \begin{cases} at & \text{if } 0 < t \le T/2 \\ a(T - t) & \text{if } T/2 \le t < T \end{cases} \tag{4}$$

   Find the response of the system $x(t)$ as a Fourier series.

5. (*15 pts*) A guitar string of length $L = 65\,\text{cm}$ is plucked by pulling it to the shape

$$y(x,0) = \begin{cases} ax^2 & \text{if } 0 < x < L/3 \\ (a/4)(L-x)^2 & \text{if } L/3 < x < L \end{cases} \tag{5}$$

and then letting go.

   (a) Determine the subsequent motion of the string (that is, find $y(x,t)$ for $t > 0$).

   (b) Which harmonics are excited? Why? (Contrast to plucking in the middle.)

   (c) Use Mathematica to plot the original string shape and then the string displacement as a function of x for $t = 0$, 0.4, and 0.8 times L/v (attach your plot). Comment on the plots.

Optional problems (counts as bonus points)

6. (*10 pts*) Develop the Fourier series for the function $f(x) = x^2$ over the range $0 \le x \le 1$ and make/attach a Mathematica plot as in problem 1.

7. (*10 pts*) Attempt to solve the equation

$$x^2 y'' + y' = 0 \tag{6}$$

using the Frobenius method. Show that the resulting series does not converge for any value of x (you may need to look up tests of convergence of a series).

8. (*5 pts*) A function $f(x)$ is expanded in an exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{7}$$

If $f(x)$ is real, $f(x) = f^*(x)$, what restriction is imposed on the coefficients c_n?