Physics 7701: Problem Set #5

The problems are due in Russell Colburn's mailbox in the main office by noon on Thursday, October 3. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*20 pts*) A single-loop series LRC circuit has resistance $R = 15\,\Omega$, inductance $L = 10\,\text{mH}$, and capacitance $C = 1.5\,\mu\text{F}$. A half-rectified sine wave power supply (see Problem 4.12 figure in Lea) with period $T = 1.57 \times 10^{-3}\,\text{s}$ is attached to the circuit. Find the voltage across the capacitor as a Fourier series in time once the circuit has reached a steady state. You are invited to use Mathematica. [Note: The solution to Lea Example 4.3, which is a similar problem, is not correct. See the 7701 web page for a corrected version.]

2. (*20 pts*) Show *using contour integration (and not Mathematica for the integrals except to check)* that the following sequences of functions are delta sequences:

   (a) $\phi_n(x) = \frac{n}{\pi}\left(\frac{1}{1+n^2x^2}\right)$

   (b) $\phi_n(x) = \frac{1-\cos nx}{n\pi x^2}$

3. (*20 pts*) A uniform rod of length l and mass M lies along the x-axis with one end at the origin. Express the density in terms of delta functions

   (a) in rectangular Cartesian coordinates

   (b) in cylindrical coordinates

   (c) in spherical coordinates

4. (*20 pts*) Evaulate

   (a) $\int_{-\infty}^{\infty} e^{-|x|}\delta(x^2+2x-3)\,dx$

   (b) $\int_{-\infty}^{\infty} e^{-x^2}\delta(x^2+x-6)\,dx$

5. (*20 pts*) A string of length L, with tension T and mass per unit length μ, is hit simultaneously at $t = 0$ at the two points $x = L/3$ and $x = 2L/3$. The impulse delivered at each point is I. Find the subsequent displacement of the string.

Optional problems (counts as bonus points)

6. (*10 pts*) Find a Fourier series representation of the delta function $\delta(x)$ in the range $(-L, +L)$ in two ways:

   (a) Start with the Fourier series for a step function and differentiate.

   (b) Start with the block functions (which form a delta sequence)

$$\phi_n(x) = \begin{cases} n/2 & \text{if } -1/n < x < 1/n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   and form the Fourier series. Take the limit as $n \to \infty$.

   (c) Are the results the same? If not, why not? Give a quantitative as well as a qualitative account of any discrepancy.

7. (*10 pts*) A rectangular box measuring $a \times b \times c$ has all its walls at temperature T_1 except for the one at $z = c$, which is held at temperature T_2. When the box comes to equilibrium, the temperature function $T(x, y, z)$ satisifes the equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T \tag{2}$$

with the time derivative on the left equal to zero. Using the method in Lea, Example 3.15, find the temperature T in the box in the form

$$T(x, y, z) = T_1 + \tau(x, y, z) \tag{3}$$

where τ is expressed in a Fourier series

$$\tau(x, y, z) = \sum_{n,m} a_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} f(z) \tag{4}$$

Find the function $f(z)$ and the coefficients a_{mn}.