Physics 7701: Problem Set #6

The problems are due in Russell Colburn's mailbox in the main office by 4pm on Tuesday, October 15. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*20 pts*) Find the Fourier transform of the following functions, and verify your results by computing the inverse transform. Do these "by hand" but check your answers with Mathematica. You can assume that parameters α, β, and a are real and positive.

   (a) $e^{-\alpha x^2}\cos\beta x$

   (b) $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

   (c) $f(t) = \begin{cases} te^{-at} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$

   (d) $\dfrac{x}{x^2 + a^2}$

2. (*15 pts*) Verify Parseval's theorem in the form

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk \tag{1}$$

   by calculating the transform of

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

   and evaluating the integrals of $|f(x)|^2$ and $|F(k)|^2$. You can use Mathematica for any part of this.

3. (*20 pts*) A spring-and-dashpot system satisfies the equation

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t) \tag{3}$$

   with $\omega_0 > \alpha > 0$. The driving force per unit mass $f(t)$ is zero for $t < 0$ and

$$f(t) = e^{-\alpha t}\sin\Omega t \tag{4}$$

   for $t > 0$. Find $x(t)$ for $t > 0$, and verify that your method gives $x = 0$ for $t < 0$.

4. (*25 pts*) Radon diffuses from the ground into the atmosphere at a rate r (atoms/m$^2\cdot$ s). Model the atmosphere as a semi-infinite medium with boundary (the ground) at $y = 0$. Then the density $\rho(y, t)$ of atmospheric radon is described by the equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial y^2} - \lambda\rho \tag{5}$$

where D is the appropriate diffusion coefficient and λ is the decay rate for radon. The boundary condition at the ground is

$$\left.\frac{\partial \rho}{\partial y}\right|_{y=0} = \text{constant} = -\alpha\ . \tag{6}$$

You are allowed (and encouraged) to use Mathematica in this problem.

(a) What is the boundary condition at $y \to \infty$? (Explain, don't just give the answer!)

(b) Derive an integral expression for $\rho(y, t)$ in the case that $\rho(y, 0) = 0$ using a Fourier cosine transform in y.

(c) Evaluate $\partial\rho/\partial t$ at $t = 0$, and hence determine α in terms of r and D.

(d) Obtain expressions for $\rho(0, t)$ and $\rho(y, \infty)$.

(e) Obtain $\rho(y, t)$ as an integral over t.

5. (*20 pts*) Find the three-dimensional Fourier transform of the charge distribution

$$\rho(\mathbf{r}) = \rho_0 \frac{e^{-r/a}}{4\pi r}\ , \tag{7}$$

which is called the form factor of a charged particle. Derive this "by hand" but use Mathematica to check your answer.

Optional problems (counts as bonus points)

6. (*10 pts*) Find the Fourier transform of the function

$$f(t) = \begin{cases} A\cos\omega_0 t & \text{if } -T < t < T \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

7. (*10 pts*) Show that the square deviation between two functions,

$$D = \int_{-\infty}^{\infty} |f(x) - g(x)|^2 dx \tag{9}$$

equals the square deviation between the transforms,

$$D = \int_{-\infty}^{\infty} |F(k) - G(k)|^2 dk \tag{10}$$