Physics 7701: Problem Set #7

These problems are due in Russell Colburn's mailbox in the main office by 4:30pm on Wednesday, October 23. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

1. (*20 pts*) **Vector operations in cylindrical and spherical coordinates.** Use the formulas on the back cover of Jackson (or equivalent) to evaluate the following expressions. In parts (a) and (b) assume that $\rho > 0$ and $r > 0$.

   (a) In cylindrical coordinates, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}}$, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}$, $\boldsymbol{\nabla}\times\hat{\boldsymbol{\rho}}$, $\boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}}$, $\boldsymbol{\nabla}\ln\rho$, $\nabla^2\ln\rho$.

   (b) In spherical coordinates, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{r}}$, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\theta}}$, $\boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}}$, $\boldsymbol{\nabla}\times(r\hat{\boldsymbol{\theta}})$, $\nabla^2(1/r)$.

   (c) [BONUS] Consider the expressions in (a) and (b) that depend explicitly on ρ or r and determine if they have a delta function at the origin. If they do, calculate its coefficient in the specified coordinates.

2. (*20 pts*) Some basic practice with vector calculus theorems (more in the bonus problems).

   (a) Evaluate the integral (using a vector calculus theorem)

   $$\oint_C d\mathbf{r}\cdot\mathbf{E} \qquad \text{where} \qquad \mathbf{E} = x^2y\hat{\boldsymbol{x}} - xy^2\hat{\boldsymbol{y}} \tag{1}$$

   and C is the unit circle in the x–y plane centered at the origin.

   (b) Evaluate the integral (using a vector calculus theorem)

   $$\int_S \mathbf{E}\cdot d\mathbf{A} \qquad \text{where} \qquad \mathbf{E} = x^2yz(\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) \tag{2}$$

   and S is a hemisphere of radius 1, with the center of the sphere at the origin, the flat side in the x–y plane.

3. (*20 pts*) **Charge distributions with delta functions.** Use Dirac delta functions in the specified coordinates to find the three-dimensional charge densities $\rho(\mathbf{x})$ corresponding to the following charge distributions (Jackson 1.3):

   (a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

   (b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disk of negligible thickness and radius R.

(d) Same as (c), but in spherical coordinates.

4. (*20 pts*) Here's a standard problem applying Gauss's theorem in a symmetric case (Jackson 1.4). For each of the three charged spheres of radius a with charge densities specified below, find the electric fields both inside and outside the sphere. Sketch the behavior of the fields as a function of radius in each case (in the third case for $n = -2$ and $n = +2$).

   (a) A conducting sphere;

   (b) a sphere with a uniform (constant) charge density in its volume;

   (c) a sphere with a spherically symmetric charge density that varies radially as r^n $(n > 3)$

5. (*20 pts*) The time-averaged electrostatic potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right) , \tag{3}$$

   where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, where a_0 is the Bohr radius. Find the distribution of charge (which has both continuous and discrete parts) that will give this potential and interpret each term physically. (From Jackson 1.5.)

Optional problems (count as bonus points)

6. (*20 pts*) Consider the three-dimensional wave equation

$$\frac{\partial^2 s}{\partial t^2} - v^2 \nabla^2 s = f(\mathbf{x}, t) \ . \tag{4}$$

   (a) Take the Fourier transform of this equation and solve for the transform $S(\mathbf{k}, \omega)$.

   (b) Show that the introduction of a damping force (through the addition of a term $\alpha\, \partial s/\partial t$ on the left-hand side) moves the poles off the real axis.

   (c) Invert the transform in the case $\alpha \to 0^+$ for $f(\mathbf{x}, t) = e^{-r/a}\delta(t)$, where r is distance from the origin and a is a positive constant.

   (d) Invert the transform in the case $\alpha \to 0^+$ for $f(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t)$.

7. (*10 pts*) More basic practice with vector calculus theorems.

   (a) Evaluate the integral (using a vector calculus theorem)

$$\oint_C d\mathbf{r} \cdot \mathbf{E} \qquad \text{where} \qquad \mathbf{E} = xy^2 \hat{\boldsymbol{x}} + yx^2 \hat{\boldsymbol{y}} \tag{5}$$

   and C is a semicircle of radius a in the x–y plane with the flat side along the x-axis, the center of the circle at the origin.

   (b) Evaluate the integral (using a vector calculus theorem)

$$\int_S \mathbf{E} \cdot d\mathbf{A} \qquad \text{where} \qquad \mathbf{E} = x^3 \hat{\boldsymbol{x}} + 3yz^2 \hat{\boldsymbol{y}} + 3y^2 z \hat{\boldsymbol{z}} \tag{6}$$

   and S is a sphere of radius 2 centered on the origin.