

## Physics 7701: Problem Set #8

These problems are due in Russell Colburn's mailbox in the main office by 4:30pm on Thursday, October 31. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

### Required problems

- (20 pts) **Simple capacitors** (Jackson 1.6). A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of
  - two concentric conducting spheres with radii  $a$ ,  $b$  ( $b > a$ );
  - two concentric conducting cylinders of length  $L$ , large compared to their radii  $a$ ,  $b$  ( $b > a$ ).
- (20 pts) **Force between conductors** (Jackson 1.9). Calculate the attractive force between conductors in the parallel plate capacitor (two large, flat, conducting sheets of area  $A$ , separated by a small distance  $d$ ) for
  - fixed charges on each conductor;
  - fixed potential difference between conductors.
- (40 pts) **Two electrostatic theorems** (Zangwill 3.10).
  - Use Green's second identity (also known as Green's theorem)

$$\int_V d^3x (f \nabla^2 g - g \nabla^2 f) = \oint_S d\mathbf{S} \cdot (f \nabla g - g \nabla f) \quad (1)$$

to prove that the potential  $\phi(0)$  at the center of a charge-free spherical volume  $V$  is equal to the average of  $\phi(\mathbf{r})$  over the surface  $S$  of the sphere.

- Use the result of part (a) to derive Earnshaw's theorem: The scalar potential  $\phi(\mathbf{x})$  in a finite, charge-free region of space  $R$  takes its maximum or minimum values on the boundary of  $R$ .
- (20 pts) **A variation on Coulomb's law** (Zangwill 2.21). Suppose that the electrostatic potential produced by a point charge was not Coulombic, but instead varied with distance in a manner determined by a specified scalar function  $f(r)$ :

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} f(|\mathbf{r}|). \quad (2)$$

- (a) Calculate the potential produced by an infinite flat sheet at  $z = 0$  with uniform charge per unit area  $\sigma$ .
- (b) Use the result of part (a) to show that the associated electric field is

$$\mathbf{E}(z) = \hat{\mathbf{z}} \frac{\sigma}{2\epsilon_0} z f(z). \quad (3)$$

**Optional problems (count as bonus points)**

5. (10 pts) **Variational principle for capacitance** (Jackson 1.17). A volume  $V$  in vacuum is bounded by a surface  $S$  consisting of several separate conducting surfaces  $S_i$ . One conductor is held at *unit* potential and all the other conductors at zero potential.

- (a) Show that the capacitance of the one conductor is given by

$$C = \epsilon_0 \int_V |\nabla\Phi|^2 d^3x \quad (4)$$

where  $\Phi(\mathbf{x})$  is the solution for the potential.

- (b) Show that the true capacitance  $C$  is always less than or equal to the quantity

$$C[\Psi] = \epsilon_0 \int_V |\nabla\Psi|^2 d^3x \quad (5)$$

where  $\Psi$  is any trial function satisfying the boundary conditions on the conductors. This is a variational principle for the capacitance that yields an *upper bound*.

6. (20 pts) **Electrostatic Green functions** (Jackson 1.14). Consider the electrostatic Green functions for Dirichlet and Neumann boundary conditions on the surface  $S$  bounding the volume  $V$ . Apply Green's theorem with integration variable  $\mathbf{y}$  and  $\phi = G(\mathbf{x}, \mathbf{y})$ ,  $\psi = G(\mathbf{x}', \mathbf{y})$ , with  $\nabla_y^2 G(\mathbf{z}, \mathbf{y}) = -4\pi\delta^3(\mathbf{y} - \mathbf{z})$ . Find an expression for the difference  $[G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}', \mathbf{x})]$  in terms of an integral over the boundary surface  $S$ .

- (a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that  $G_D(\mathbf{x} - \mathbf{x}')$  must be symmetric in  $\mathbf{x}$  and  $\mathbf{x}'$ .
- (b) For Neumann boundary conditions, use the boundary condition for  $G_N(\mathbf{x}, \mathbf{x}')$ :

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{S} \quad \text{for } \mathbf{x}' \text{ on } S, \quad (6)$$

where  $S$  is the total area of the boundary surface, to show that  $G_N(\mathbf{x}, \mathbf{x}')$  is not symmetric in general, but that  $G_N(\mathbf{x}, \mathbf{x}') - F(\mathbf{x})$  is symmetric in  $\mathbf{x}$  and  $\mathbf{x}'$ , where

$$F(\mathbf{x}) = \frac{1}{S} \oint_S G_N(\mathbf{x}, \mathbf{y}) da_y \quad (7)$$

- (c) Show that the addition of  $F(\mathbf{x})$  to the Green function does not affect the potential  $\Phi(\mathbf{x})$ .