

Physics 7701: Problem Set #9

These problems are due in Russell Colburn's mailbox in the main office by 4:30pm on Wednesday, November 13. Check the 7701 webpage for suggestions and hints. Please give feedback early and often (and email or stop by M2048 to ask about anything).

There are two groups of problems. The first group is required of everyone. The second group is optional but is recommended to go into greater depth in the material, if you have time. These will be awarded bonus points.

Required problems

- (20 pts) **Method of images** (Jackson 2.2). Using the method of images, discuss the problem of a point charge q *inside* a hollow, grounded, conducting sphere of inner radius a . Find
 - the potential inside the sphere;
 - the induced surface-charge density;
 - the magnitude and direction of the force acting on q .
 - Is there any change in the solution if the sphere is kept at a fixed potential V ? If the sphere has a total charge Q on its inner and outer surfaces?
- (20 pts) **The potential of a voltage patch** (Zangwill 8.15). The plane $z = 0$ is grounded except for a finite area S_0 which is held at potential ϕ_0 . Show that the electrostatic potential away from the plane is

$$\phi(x, y, z) = \frac{\phi_0 |z|}{2\pi} \int_{S_0} \frac{d^2 r'}{|\mathbf{r} - \mathbf{r}'|^3} \quad (1)$$

- (20 pts) **Green function expansion** (Jackson 2.15).
 - Show that the Green function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \leq x \leq 1$, $0 \leq y \leq 1$, has an expansion

$$G(x, y; x', y') = 2 \sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x') \quad (2)$$

where $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2 \pi^2 \right) g_n(y, y') = -4\pi \delta(y' - y) \quad \text{and} \quad g_n(y, 0) = g_n(y, 1) = 0. \quad (3)$$

- Taking for $g_n(y, y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x, y; x', y') = 8 \sum_{n=1}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh(n\pi y_{<}) \sinh(n\pi(1 - y_{>})) \quad (4)$$

where $y_{<}$ ($y_{>}$) is the smaller (larger) of y and y' .

4. (20 pts) **Applying a Green function expansion** (Jackson 2.16). A two-dimensional potential exists on a unit square area ($0 \leq x \leq 1, 0 \leq y \leq 1$) bounded by “surfaces” held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z). Using the Green function of the previous problem, show that the solution can be written as

$$\Phi(x, y) = \frac{4}{\pi^3 \epsilon_0} \sum_{m=0}^{\infty} \frac{\sin[(2m+1)\pi x]}{(2m+1)^3} \left\{ 1 - \frac{\cosh[(2m+1)\pi(y-1/2)]}{\cosh[(2m+1)\pi/2]} \right\} \quad (5)$$

5. (20 pts) **The mysterious hollow cube** (Jackson 2.23). A hollow cube has conducting walls defined by six planes $x = 0, y = 0, z = 0$, and $x = a, y = a, z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V . The other four sides are at zero potential.
- Find the potential $\Phi(x, y, z)$ at any point inside the cube.
 - Evaluate the potential at the center of the cube numerically using Mathematica, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy. Compare your numerical result with the average value of the potential on the walls.
 - (Bonus) Prove that the potential at the center of a cube with conducting walls is the average of the potential on the walls. (This is actually true for any closed polyhedron!)

Optional problems (count as bonus points)

6. (10 pts) **More image charges** (Jackson 2.3). A straight-line charge with constant linear charge density λ is located perpendicular to the x - y plane in the first quadrant at (x_0, y_0) . The intersecting planes $x = 0, y \geq 0$ and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.
- The well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x, y) = (\lambda/4\pi\epsilon_0) \ln(R^2/r^2)$, where $r^2 = (x - x_0)^2 + (y - y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.
 - Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot using Mathematica σ/λ versus x for $(x_0 = 2, y_0 = 1), (x_0, y_0 = 1)$, and $(x_0 = 1, y_0 = 2)$.
7. (10 pts) **Using a cube to simulate a point charge** (Zangwill 8.19). Use the Dirichlet Green function from class (shifted to have its center at the origin) to find the charge density that must be glued onto the surfaces of an insulating box with side walls at $x = \pm a, y = \pm a$, and $z = \pm a$ so that the electric field everywhere outside the box is identical to the field of a (fictitious) point charge Q located at the center of the box. It is sufficient to calculate $\sigma(x, y)$ for the $z = a$ face. Give a numerical value (accurate to 0.1%) for $\sigma(0, 0)$.