Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi\mathbf{a}) = \mathbf{a} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi\mathbf{a}) = \nabla\psi \times \mathbf{a} + \psi\nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla \cdot \mathbf{x} = 3 \qquad\qquad \nabla \times \mathbf{x} = 0$$

$$\nabla \cdot [\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla \times [\mathbf{n}f(r)] = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})] + \mathbf{n}(\mathbf{a} \cdot \mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + i(\mathbf{L} \times \mathbf{a})$$

where $\mathbf{L} = \dfrac{1}{i}(\mathbf{x} \times \nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A} \, d^3x = \int_S \mathbf{A} \cdot \mathbf{n} \, da \qquad\qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi \, d^3x = \int_S \psi\mathbf{n} \, da$$

$$\int_V \nabla \times \mathbf{A} \, d^3x = \int_S \mathbf{n} \times \mathbf{A} \, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi) \, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi \, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi) \, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n} \, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad\qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi \, da = \oint_C \psi \, d\mathbf{l}$$

Where to Find Key Material on Special Functions

Explicit Forms of Vector Operations

Let e_1, e_2, e_3 be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of A. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla \psi = e_1 \frac{\partial \psi}{\partial x_1} + e_2 \frac{\partial \psi}{\partial x_2} + e_3 \frac{\partial \psi}{\partial x_3}$$

$$\nabla \cdot A = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times A = e_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + e_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + e_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla \psi = e_1 \frac{\partial \psi}{\partial \rho} + e_2 \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} + e_3 \frac{\partial \psi}{\partial z}$$

$$\nabla \cdot A = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_1) + \frac{1}{\rho} \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times A = e_1 \left(\frac{1}{\rho} \frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z} \right) + e_2 \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho} \right) + e_3 \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_2) - \frac{\partial A_1}{\partial \phi} \right)$$

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla \psi = e_1 \frac{\partial \psi}{\partial r} + e_2 \frac{1}{r} \frac{\partial \psi}{\partial \theta} + e_3 \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi}$$

$$\nabla \cdot A = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_2) + \frac{1}{r \sin \theta} \frac{\partial A_3}{\partial \phi}$$

$$\nabla \times A = e_1 \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_3) - \frac{\partial A_2}{\partial \phi} \right]$$
$$+ e_2 \left[\frac{1}{r \sin \theta} \frac{\partial A_1}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right] + e_3 \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial \theta} \right]$$

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \psi). \right]$$