Physics 880.05: Problem Set #1

These problems are due in class on Wednesday, January 15. All of the problems build on the dilute Fermi gas with δ–function interaction $V(\mathbf{x}-\mathbf{x}') = \lambda\,\delta^3(\mathbf{x}-\mathbf{x}')$ that was introduced in class (see the online lecture notes). Check the 880.05 webpage for suggestions and hints. Please give feedback early and often.

1. Repeat the calculations done in class but for a Fermi "fluid" in *one* spatial dimension with two-body interaction $V(x-x') = \lambda\delta(x-x')$. [Note: For an attractive δ function ($\lambda < 0$), this system can be solved to all orders numerically in terms of some relatively simple coupled integral equations.]

   (a) When do you expect perturbation theory to be a good approximation? (I.e., low density or high density or never?)

   (b) Find the ground-state energy to first-order in the interaction.

   (c) Compare the one- and three-dimensional cases qualitatively for both attractive and repulsive interactions.

2. We can consider the non-interacting Fermi gas ground state $|F\rangle$ as a *trial wave function*. Thus, when we take the expectation value of the full Hamiltonian in this state, we are doing a variational calculation. Let's consider one of the implications.

   (a) By considering the resulting energy per particle $(E^{(0)} + E^{(1)})/N$ as a function of density for an attractive delta function interaction in three dimensions, prove that the *exact* system must collapse. [Note: We're ignoring the subtleties of the delta function interaction in three dimensions, but the result is generally true for an attractive, short-range interaction.]

   (b) Repeat the analysis for the one-dimensional system. Does it have to collapse?

3. Second-order perturbation theory for the dilute Fermi gas. If $H = H_0 + H_1$ and the unperturbed eigenvectors $|j\rangle$ satisfy $H_0|j\rangle = E_j|j\rangle$, then

$$E^{(2)} = \sum_{j\neq 0} \frac{|\langle 0|H_1|j\rangle|^2}{E_0 - E_j} = \langle 0|H_1 \frac{P}{E_0 - H_0} H_1|0\rangle$$

   where $|0\rangle$ is the ground-state eigenvector of H_0 with energy E_0, and $P = 1 - |0\rangle\langle 0|$ is a projection operator on the excited states [*from* $F{+}W$].

(a) Show that the second-order contribution to the ground-state energy for a δ-function interaction in three-dimensions *diverges* (i.e., is infinite). Which intermediate states are the problem?

(b) Repeat the analysis in one dimension. Why is there a difference?

4. (BONUS) [F+W 1.7] Consider a polarized dilute spin–1/2 Fermi gas (δ–function interaction) in which $N_\pm$ denotes the number of fermions with spin-up (down).

   (a) Find the ground-state energy to first order in the interaction potential as a function of $N = N_+ + N_-$ and the polarization $\zeta = (N_+ - N_-)/N$.

   (b) Show that the system is partially "magnetized" for

$$20/9 < \frac{\lambda N}{\Omega \overline{T}} < (5/3)2^{2/3} ,$$

   where $\overline{T}$ is the mean kinetic energy per particle in the unmagnetized state, and N/Ω is the corresponding particle density.

   (c) What happens outside of these limits? (Explain the *physics*.)