

Physics 880.05: Problem Set #2

These problems are due on Wednesday, February 6. See the online lecture notes for details of any of the individual topics covered in this problem set. Check the 880.05 webpage for suggestions and hints. Please give feedback early and often.

- [N \mathcal{E} O 1.7] Stability and Equilibrium Conditions for Uniform Matter.** Consider the thermodynamic limit of a system of N particles at zero temperature confined in a box of volume Ω in a state with energy E having uniform density. Denote the density $\rho = N/\Omega$ and the energy per particle $\epsilon(\rho) = E/N$.
 - Show that the pressure is given by $P = \rho^2 \frac{\partial \epsilon}{\partial \rho}$. Show that the equilibrium density, often called the saturation density, for self-bound unconfined bulk matter occurs at the density that minimizes $\epsilon(\rho)$. [See lecture notes!]
 - Show that the chemical potential $\mu = E(N+1) - E(N)$ is given by $\mu = \epsilon + P/\rho$. Thus, for a self-bound saturating system at equilibrium density $\mu = \epsilon$.
 - Find the saturation energy and density (in first-order perturbation theory) for the one-dimensional system considered in the first problem set.
 - Examine the stability with respect to long wavelength density fluctuations as follows. Consider dividing the volume Ω in half, and calculate the total energy if one half has density $\rho + \delta$ and the other half has density $\rho - \delta$. Show that this energy exceeds that of the uniform configuration, and thus the system is stable, if $\frac{\partial^2}{\partial \rho^2}(\rho\epsilon(\rho)) > 0$, which is equivalent to $\partial P/\partial \rho > 0$. Is the one-dimensional system stable to such fluctuations?
- [F \mathcal{E} W 2.7] Nuclei as a Fermi gas.** As a first approximation to atomic nuclei, consider the nucleus as a degenerate ($T = 0$) *noninteracting* Fermi gas of neutrons and protons.
 - What is the degeneracy factor for each level?
 - If the radius of a nucleus with A nucleons is given by $R = r_0 A^{1/3}$ with $r_0 \approx 1.2 \times 10^{-13}$ cm, what are k_F and ϵ_F ? How do they vary with A ?
 - What is the pressure exerted by this Fermi gas? Why is there pressure?
 - If each nucleon is considered to be moving in a constant potential of depth V_0 , how large must V_0 be?
 - What is the scale of the temperature at which the nucleus will act like a collection of particles described by Boltzmann statistics (i.e., when are Fermi statistics irrelevant)?

3. **Symmetry Factors.** Apply the Feynman rules for the model partition function discussed in class, $Z = \int dx e^{-ax^2/2 - \lambda x^4/4}$, to derive an expression for the λ^3 contribution to $\langle x^2 \rangle$.

- (a) Draw all of the distinct diagrams that contribute. Explain how you know what type of diagram to draw (e.g., number of vertices, connected vs. disconnected, etc.).
- (b) Find the symmetry factor for each diagram.
- (c) Use the symmetry factors and the other Feynman rules to find an expression for $\langle x^2 \rangle$ in terms of a and λ .

4. **Using Second-quantized Field Operators.**

- (a) Prove that the number operator $\hat{N} = \int \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) d^3x$ commutes with a Hamiltonian \hat{H} that has a one-body term and a two-body potential.
- (b) Find the second-quantized Hamiltonian in terms of fields when there is both a two-body potential $v_2(\mathbf{x}_i - \mathbf{x}_j) = -\lambda \delta^3(\mathbf{x}_i - \mathbf{x}_j)$ and a (repulsive) three-body potential $v_3(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k) = \beta \delta^3(\mathbf{x}_i - \mathbf{x}_j) \delta^3(\mathbf{x}_i - \mathbf{x}_k)$.

5. **Path integral for quantum mechanics \implies Schrödinger Equation.**

The composition rule for transition amplitudes is

$$U(x_f t_f; x_i t_i) = \int dy U(x_f t_f; y t_m) U(y t_m; x_i t_i) . \quad (1)$$

Keeping the initial space-time point $x_i t_i$ fixed, the composition rule can be applied to find the evolution of the wave function $\psi(x, t)$ upon identifying

$$\psi(x, t) \equiv U(x t; x_i t_i) . \quad (2)$$

Therefore, for a small time evolution, $t_i = t$ and $t_f = t + \varepsilon$, we have

$$\psi(x, t + \varepsilon) = \int dy U(x t + \varepsilon; y t) \psi(y, t) . \quad (3)$$

One can recover the Schrödinger equation for $\psi(x, t)$ by using Feynman's path integral hypothesis in this expression for the infinitesimal time evolution U ,

$$U(x t + \varepsilon; y t) = \frac{1}{A} \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2} \frac{(x - y)^2}{\varepsilon} - \varepsilon V \left(\frac{x + y}{2} \right) \right] \right\} . \quad (4)$$

- (a) Prove the composition rule.
- (b) Argue that the contributions to the integral originate from $|x - y| \lesssim \sqrt{\varepsilon}$ and thus one can write $y = x + \eta$, where η is order $\sqrt{\varepsilon}$. Show that A is determined from the zeroth-order expansion in η and ε .
- (c) Argue that the odd terms in η don't contribute when integrated.
- (d) Interpret the equation you obtain at order ε .