Physics 880.05: Problem Set #3

These problems are due on Wednesday, February 19. See the online lecture notes for details of any of the individual topics covered in this problem set. Check the 880.05 webpage for suggestions and hints. Please give feedback early and often.

1. **Directly Solving for $\mathcal{G}^0$.** Your goal is to solve

$$\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2M} - \mu\right)\mathcal{G}^0(\mathbf{x}\tau;\mathbf{x}'\tau') = \delta^3(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')$$

for $\mathcal{G}^0$, subject to the fermion finite temperature boundary conditions:

$$\mathcal{G}^0(\mathbf{x}\beta;\mathbf{x}'\tau') = -\mathcal{G}^0(\mathbf{x}0;\mathbf{x}'\tau') \ .$$

(a) First go to momentum space. Introduce the Fourier transform:

$$\begin{aligned}
\mathcal{G}^0(\mathbf{k};\tau-\tau') &= \int d^3x\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,\mathcal{G}^0(\mathbf{x}\tau;\mathbf{x}'\tau') \\
\mathcal{G}^0(\mathbf{x}\tau;\mathbf{x}'\tau') &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,\mathcal{G}^0(\mathbf{k};\tau-\tau')
\end{aligned}$$

and find what differential equation $\mathcal{G}^0(\mathbf{k};\tau-\tau')$ satisfies. How do we know that $\mathcal{G}^0$ is a function of $(\mathbf{x}-\mathbf{x}')$ and $(\tau-\tau')$?

(b) Split the time into two regions: $(\tau-\tau') > 0$ and $(\tau-\tau') < 0$. Find general solutions to the differential equation in each region (they will have different coefficients!).

(c) What are the matching conditions for the two solutions? Determine the coefficients by matching and applying the boundary condition, which then gives you the solution to $\mathcal{G}^0(\mathbf{x}\tau;\mathbf{x}'\tau')$. Write it with appropriate θ-functions.

2. **The Beachball Diagram.** There are two diagrams at second order in the expansion of the energy density at $T = 0$. For a uniform system, the non-vanishing diagram is called the "beachball" diagram.

(a) An expression for the energy density in three dimensions is derived in the notes in coordinate space. Show that this is the same as second-order perturbation theory and identify the divergence found earlier.

(b) Reproduce this same expression by changing variables and carrying out the frequency integrals in the *momentum space* version (see notes).

(c) Find and evaluate (all the way!) the second-order energy density in one dimension. Compare to the first-order result and sketch as a function of density for attractive and repulsive potentials.

3. **Vanishing Diagrams at Third Order.** Consider the third-order contributions to the energy density at $T = 0$ for a delta-function potential.

   (a) Write an expression based on the Feynman rules for each one. (Don't evaluate any integrals!)

   (b) Evaluate the spin sum for each diagram. (You can use the Mathematica notebook.)

   (c) Which diagrams are equal to zero (because they are anomalous or for other reasons)?

4. **Three-body forces.** Consider a three-body potential as in Problem Set 2.

   (a) What is the lowest-order diagram with this potential?

   (b) Evaluate the contribution to the energy density at $T = 0$.

5. **Evaluation of the Grand Potential from the One-Particle Green's Function** [*N&O 5.1*]**.** Your goal is to show that

$$\Omega - \Omega_0 = -\frac{1}{2}\int_0^1 \frac{d\alpha}{\alpha}\int d^3x\, d^3x'\, d\tau\, \Sigma^\alpha(\mathbf{x}\tau;\mathbf{x}'\tau')\mathcal{G}^\alpha(\mathbf{x}'\tau';\mathbf{x}\tau^+)$$

   reproduces the diagrammatic expansion for $\Omega - \Omega_0$ through third order. [*Details to follow in class!*]