Physics 880.05: Problem Set #4

These problems are designed to review and reinforce topics from last quarter, which continue this quarter. They are due in two weeks (April 14). See the online lecture notes for details of any of the individual topics covered in this problem set. Check the 880.05 webpage for suggestions and hints. Please give feedback early and often.

1. **Skyrme-type model for nuclear matter.** As a model for nuclear matter, combine an attractive two-body delta function force proportional to λ and a repulsive three-body contact force proportional to β (as in the third problem set), evaluated at the Hartree-Fock level (leading diagrams in each case) for degeneracy 4 (neutrons plus protons) in three dimensions.

   (a) Find values of λ and β such that the minimum in the energy per particle $\epsilon(\rho)$ occurs at the experimental values $\rho_0 = 0.16\,\mathrm{fm}^{-3}$ and $\epsilon(\rho_0) = -16\,\mathrm{MeV}$. (Don't forget the kinetic energy!)

   (b) Plot $\epsilon(\rho)$ and identify regions for which it has positive pressure and for which it is stable against long wavelength density fluctuations.

   (c) Calculate the compression modulus $\kappa \equiv k_F^2 \frac{\partial^2 \epsilon}{\partial k_F^2}$ at the equilibrium Fermi momentum. Compare to the corresponding value for a noninteracting Fermi gas at the same density. The "experimental" value has been disputed; it is typically said to be between 200 MeV and 300 MeV. Bonus: How might you measure κ in a finite nucleus?

2. **Lehmann representation for advanced and retarded functions.** It is often convenient to deal with functions that are analytic in the upper or lower half plane. Thus we define the retarded and adanced Green's functions:

$$
\begin{aligned}
iG^R(\mathbf{x}t, \mathbf{x}'t') &= \langle\Psi_0|\{\widehat{\Psi}_{H\alpha}(\mathbf{x}t), \widehat{\Psi}^\dagger_{H\beta}(\mathbf{x}'t')\}|\Psi_0\rangle\theta(t_1 - t_1') \ , \\
iG^A(\mathbf{x}t, \mathbf{x}'t') &= \langle\Psi_0|\{\widehat{\Psi}_{H\alpha}(\mathbf{x}t), \widehat{\Psi}^\dagger_{H\beta}(\mathbf{x}'t')\}|\Psi_0\rangle\theta(t_1' - t_1) \ ,
\end{aligned}
$$

   where the $\{\}$'s denote an anticommutator. For either one of these functions (your choice), derive the Lehmann representation analogous to the one on page 174 of the class notes.

3. **Spin-dependent force.** Revisit the diagrammatic calculation of the energy density in three-dimensions through second order, but now with a spin-dependent potential:

$$
V_s(\mathbf{x}_1, \mathbf{x}_2)_{\alpha\beta,\lambda\mu} = \lambda_s\, \vec{\sigma}(1)_{\alpha\beta} \cdot \vec{\sigma}(2)_{\lambda\mu}\, \delta^3(\mathbf{x}_1 - \mathbf{x}_2) \ ,
$$

   where the spin indices have been given explicitly.

(a) What is the corresponding interaction term in the Lagrangian?

(b) What is the Feynman rule for this vertex?

(c) Calculate the first-order correction to the energy density for a dilute Fermi gas with this interaction (bow-tie diagram).

(d) Bonus: Calculate the second-order correction to the energy density for a dilute Fermi gas with this interaction. Deos the anomalous graph still vanish?

4. **Fermi liquid theory in one spatial dimension.** Consider a Fermi liquid in one spatial dimension, in which case the Fermi surface is comprised of two points, $k = \pm k_F$. The quasiparticle interaction in any spin channel is characterized by two values, $f(k_F\sigma, k_F\sigma')$ and $f(k_F\sigma, -k_F\sigma')$, and it is convenient to define

$$\begin{aligned} f(k\sigma, k'\sigma') &= f(k,k') + 4\vec{\sigma}\cdot\vec{\sigma}'\phi(k,k') \\ F_{0,1} &= N(0)\frac{1}{2}\big(f(k_F,k_F) \pm f(k_F,-k_F)\big) \\ Z_{0,1} &= N(0)\frac{1}{2}\big(\phi(k_F,k_F) \pm \phi(k_F,-k_F)\big) \end{aligned}$$

where the one-dimensional density of states is $N(0) = \frac{g}{\pi}\frac{m^*}{k_F}$ and σ represents some internal symmetry such as spin or isospin with degeneracy g. Derive the following results for

(a) the effective mass

$$\frac{m^*}{m} = 1 + F_0 \ ,$$

(b) the specific heat

$$c_V = \frac{C_V}{L} = \frac{\pi k_B^2 T m^*}{3k_F} \ ,$$

(c) the sound velocity

$$c^2 = \frac{k_F^2(1+F_0)}{m\,m^*} \ ,$$

(d) and the forward scattering sum rule

$$\sum_{i=0}^{1}\left(\frac{F_i}{1+F_i} + \frac{Z_i}{1+Z_i}\right) = 0 \ .$$