Physics 880.05: Problem Set #5

These problems are designed to review and reinforce the material we have covered in class. Please ask questions! They are due May 12 (or thereabouts). See the online lecture notes for details of any of the individual topics covered in this problem set. Check the 880.05 webpage for suggestions and hints. Please give feedback early and often.

1. **Effective action in one dimension.** The idea here is to go through our treatment of the large-N expansion of the effective action and to evaluate it in the Bose limit, but in one spatial dimension.

   (a) Derive the expression for $\mathcal{E}_{\mathrm{LO}}$ corresponding to the one on page 236 of the notes.

   (b) Derive expressions for $\mathcal{E}_{\mathrm{NLO}}$ and for $\Pi_0(q_0, q)$ in particular, corresponding to the ones on page 238. BONUS: Evaluate the integral in $\Pi_0(q_0, q)$.

   (c) Take the Bose limit and derive the LO and NLO terms for the energy density of a dilute Bose gas with a repulsive interaction. What happened to the kinetic energy?

2. **Number fluctuations in the BCS ground state.**

   (a) Show that the particle-number fluctuation in the BCS ground state $|\mathrm{BCS}\rangle$ is

   $$\frac{\langle \widehat{N}^2 \rangle - \langle \widehat{N} \rangle^2}{\langle \widehat{N} \rangle^2} = \frac{\sum_{\mathbf{k}}(u_k v_k)^2}{\left(\sum_{\mathbf{k}} v_k^2\right)^2} \ .$$

   (b) Compare the result for the BCS ground state to the normal ground state $|\mathbf{F}\rangle$.

   (c) What does it mean that the fluctuation is not zero if we calculate ^{202}Pb? BONUS: *Estimate* the mean-square fluctuation in ^{202}Pb.

3. **Another BCS ground state problem.**

   (a) Show that $a_{\mathbf{k}\uparrow}^{\dagger}|\mathrm{BCS}\rangle$ and $a_{-\mathbf{k}\downarrow}|\mathrm{BCS}\rangle$ are the same state, up to a normalization factor, and that they are orthogonal to $|\mathrm{BCS}\rangle$.

   (b) Show from the expectation value of $\widehat{K}$ that Ω is increased by E_k for these states relative to $|\mathrm{BCS}\rangle$.

4. **Feynman-Hellmann proof.** On page 305 of the class notes, we used the result:

   $$\frac{\delta \epsilon_{\alpha}}{\delta \sigma_c(\mathbf{x})} = C_0 \phi_{\alpha}^{\dagger}(\mathbf{x}) \phi_{\alpha}(\mathbf{x}) \ .$$

   (a) Derive this result.

   (b) Why is $\delta\phi_{\alpha}(\mathbf{x}')/\delta\sigma_c(\mathbf{x})$ not zero? So why do such terms not contribute to the result?