Appendix A

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A-1

Details of Euclidean Fermion path integral for $\beta \neq 0$

Start with

$$Z = \int \mathcal{D}[\psi^\dagger(x)\psi(x)]\, e^{-\int_0^{\beta}d\tau\int d^3x\, \psi^\dagger_\alpha(x)\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\psi_\alpha(x)} \times e^{-\frac{\lambda}{2}\int_0^{\beta}d\tau\int d^3x\, \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)}$$

Here $\lambda \equiv C_0$

where $x \equiv (\vec{x}, \tau)$ and $\psi^\dagger_\alpha\psi_\alpha \equiv \sum_{\alpha=1}^{\nu}\psi^\dagger_\alpha\psi_\alpha$ (implied summation)
and we're working with $\hbar = 1$ as usual

- We'll deal with Fermions only here, so the ψ's and $\psi^\dagger$'s are Grassmann functions (think of them as Grassmann variables ψ_{ijkl}, $\psi^\dagger_{ijkl}$ on a spacetime lattice with i,j,k,l corresponding to discrete values of $\tau, x, y,$ and z).
  - Grassman variables always anti-commute: $\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha$
- Do our "usual" procedure:

① generalize Z to include "sources" $\eta(x)$ and $\eta^\dagger(x)$ (Grassman functions) coupled to $\psi^\dagger$ and ψ, respectively.
- These are Grassmann sources, with spin indices

$$Z[\eta_\alpha, \eta^\dagger_\beta] = \int \mathcal{D}[\psi^\dagger\psi]\, e^{-\left(S_E + \int_0^{\beta}d\tau\int d^3x\, \eta^\dagger_\alpha(x)\psi_\alpha(x) + \psi^\dagger_\alpha(x)\eta_\alpha(x)\right)}$$

where S_E is the Euclidean action (including the chemical potential)

$$S_E = \int_0^{\beta}d\tau\int d^3x \left\{\psi^\dagger_\alpha(x)\underbrace{\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)}_{\equiv\, \mathcal{G}^{0\,-1}}\psi_\alpha(x) + \frac{\lambda}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)\right\}$$

② Remove the interaction term in favor of functional derivatives with respect to the Grassmann sources: $\psi \to \frac{\delta}{\delta\eta^\dagger}$, $\psi^\dagger \to \frac{\delta}{\delta\eta}$

$$Z[\eta_\alpha, \eta^\dagger_\alpha] = e^{-\int_0^{\beta}d\tau\int d^3x\, \frac{\lambda}{2}\left(\frac{\delta}{\delta\eta_\alpha(x)}\right)\left(\frac{\delta}{\delta\eta_\beta(x)}\right)\left(\frac{-\delta}{\delta\eta^\dagger_\beta(x)}\right)\left(\frac{-\delta}{\delta\eta^\dagger_\alpha(x)}\right)} \int \mathcal{D}[\psi^\dagger\psi]\, e^{-\int_0^{\beta}d\tau\int d^3x\left[\psi^\dagger_\alpha(x)\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\psi_\alpha(x) + \eta^\dagger_\alpha(x)\psi_\alpha(x) + \psi^\dagger_\alpha(x)\eta_\alpha(x)\right]}$$

- we're using $\boxed{\frac{\delta}{\delta\eta_\alpha(x)}e^{-\int\psi^\dagger_\beta\eta_\beta} = e^{-\int\psi^\dagger_\beta\eta_\beta}\,\psi^\dagger_\alpha(x)}$ and $\boxed{\frac{-\delta}{\delta\eta^\dagger_\alpha(x)}e^{-\int\eta^\dagger_\beta\psi_\beta} = -e^{-\int\eta^\dagger_\beta\psi_\beta}\,\psi_\alpha(x)}$

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- We make our lives easier here by moving the Grassmann fields brought down by the derivatives all the way to the right. We can do this since all of the Grassmann variables in the exponents appear in pairs. (so minus signs cancel)

③ Complete the square in the remaining path integral

- We'll carry this out with "schematic" notation, which means we'll drop the explicit $x=(\vec{x},\tau)$ indices. Since we will keep spin indices, one could also imagine those to stand for discrete time and space indices as well

$$\boxed{\begin{aligned}\Rightarrow &\int \mathcal{D}(\psi^\dagger\psi)\, e^{\int \psi^\dagger_\alpha \mathcal{G}^{0-1}_{\alpha\beta}\psi_\beta + \eta^\dagger_\alpha\psi_\alpha + \psi^\dagger_\alpha\eta_\alpha} \\ &= \int \mathcal{D}(\psi^\dagger\psi)\, e^{-\int(\psi^\dagger_\alpha + \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\alpha})\mathcal{G}^{0-1}_{\alpha\beta}(\psi_\beta + \mathcal{G}^0_{\beta\delta}\eta_\delta)}\, e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta}\eta_\delta} \\ &= e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta}\eta_\delta} \int \mathcal{D}(\psi^{\dagger\prime}\psi')\, e^{\int \psi'^\dagger_\alpha \mathcal{G}^{0-1}_{\alpha\beta}\psi'_\beta} \\ &= e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta}\eta_\delta}\, Z_0\end{aligned}}$$

- We get the third line by shifting variables — the "Jacobian" is 1 — to $\psi'_\alpha = \psi_\alpha + \mathcal{G}^0_{\alpha\beta}\eta_\beta$.
 A proof that we can change Grassmann variables like this is given in Negele and Orland, but it should be plausible from simple examples like: $\int d\psi^\dagger d\psi\, (\psi^\dagger+\eta)(\psi^\dagger+\eta^\dagger) = \int d\psi^\dagger d\psi\, \psi^\dagger\psi$.

- We've introduced $\mathcal{G}^0_{\alpha\beta}$ as the inverse of

$$\boxed{\mathcal{G}^{0-1}_{\alpha\beta} \Rightarrow \delta_{\alpha\beta}\left(\partial/\partial\tau - \frac{\nabla^2}{2m} - \mu\right)}$$

with anti-periodic boundary conditions.

④ Now we can do perturbation theory in powers of λ as usual.

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- In order to do perturbation theory, we need the non-interacting single-particle Green's function $\mathcal{G}^0_{\alpha\beta}$.

- We will follow the sign convention in Negele and Orland (which is opposite to that in Fetter and Walecka).

- Then $\mathcal{G}^0_{\alpha\beta}$ is the solution to $\mathcal{G}^{0-1}\mathcal{G}^0 = 1$, which becomes

$$\boxed{\delta_{\alpha\beta}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau;\vec{x}'\tau') = \delta_{\alpha\gamma}\,\delta^3(\vec{x}-\vec{x}')\,\delta(\tau-\tau')}$$

with the boundary condition $\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}0;\vec{x}'\tau') = -\mathcal{G}_{\alpha\beta}(\vec{x}\beta;\vec{x}'\tau')}$
(note: β sum is implied).

- There are several different ways we can derive $\mathcal{G}^0$. One of the simplest is to "guess" the answer and verify that it works. We guess

$$\boxed{\begin{aligned}\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau;\vec{x}'\tau') &= \delta_{\beta\gamma}\frac{1}{\Omega}\sum_{k} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')} \\ &\quad\times\left[\theta(\tau-\tau'-\eta)(1-n^0_k) - \theta(\tau'-\tau+\eta)\,n^0_k\right] \\ &\rightarrow \delta_{\beta\gamma}\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')}\left[\theta(\tau-\tau'-\eta)(1-n^0_k) - \theta(\tau'-\tau+\eta)\,n^0_k\right]\end{aligned}}$$

- The corresponding function for bosons has $1-n^0_k \to 1+n^0_k$ and a plus sign between the two θ-functions.

* - The η is an infinitesimal that indicates that if $\tau=\tau'$, we should keep the second term.
  - This prescription follows from a careful evaluation of $\mathcal{G}^0$ as the inverse of the discrete version of $\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)$. See Negele and Orland section 2.2 for details.
  - We take $\eta \to 0$ as soon as we've used it to pick out which θ function to take at $\tau=\tau'$.

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Let's check that it works:

i) boundary condition

$$\boxed{\mathcal{G}^0_{\beta\gamma}(\vec{x}\beta;\vec{x}'\tau') = \delta_{\beta\gamma}\frac{1}{\Omega}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)\beta}\, e^{(\epsilon^0_k-\mu)\tau'} \times (1-n^0_k)}$$

since the first θ function will be satisfied for $0 \le \tau' < \beta$.

$$\boxed{\mathcal{G}^0_{\beta\gamma}(\vec{x}0;\vec{x}'\tau') = \delta_{\beta\gamma}\frac{1}{\Omega}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{(\epsilon^0_k-\mu)\tau'}(-n^0_k)}$$

But $e^{-\beta(\epsilon^0_k-\mu)}(1-n^0_k) = e^{-\beta(\epsilon^0_k-\mu)}\left(1-\frac{1}{e^{\beta(\epsilon^0_k-\mu)}+1}\right)$

$$= \cancel{e^{-\beta(\epsilon^0_k-\mu)}}\left(\frac{\cancel{e^{\beta(\epsilon^0_k-\mu)}}}{e^{\beta(\epsilon^0_k-\mu)}+1}\right) = n^0_k$$

$$\Rightarrow \boxed{\mathcal{G}^0_{\beta\gamma}(\vec{x}\beta;\vec{x}'\tau') = -\mathcal{G}^0_{\beta\gamma}(\vec{x}0;\vec{x}'\tau')} \quad \text{as expected.}$$

ii) satisfies the differential equation:

$$\delta_{\alpha\beta}\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau;\vec{x}'\tau')$$

$$= (\delta_{\alpha\beta}\delta_{\beta\gamma})\frac{1}{\Omega}\sum_{\vec{k}}\underbrace{\left(-(\epsilon^0_k-\cancel{\mu})+\frac{k^2}{2m}-\mu\right)}_{0} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')}$$

$$+\delta_{\alpha\beta}\delta_{\beta\gamma}\frac{1}{\Omega}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')}\Big[\underbrace{\delta(\tau-\tau')(1-n^0_k)+\delta(\tau-\tau')n^0_k}_{\delta(\tau-\tau')}\Big]$$

$$= \delta_{\alpha\gamma}\,\delta(\tau-\tau')\frac{1}{\Omega}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= \delta_{\alpha\gamma}\,\delta^3(\vec{x}-\vec{x}')\,\delta(\tau-\tau') \quad \checkmark$$

where we've used $\boxed{\frac{\partial}{\partial\tau}\theta(\tau-\tau') = \delta(\tau-\tau')}$ and $\boxed{\frac{\partial}{\partial\tau}\theta(\tau'-\tau) = -\delta(\tau-\tau')}$

- You can also simply <u>derive</u> the result for $\mathcal{G}^0$ by solving the differential equation explicitly. (Use Fourier transform in $\vec{x}$ and "division of regions" methods)
- For completeness, we include yet another derivation using <u>field operators</u>

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- We define the single-particle Green's function as

$$\boxed{\begin{aligned}\mathcal{G}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') &= \frac{\int \mathcal{D}(\psi^\dagger\psi)\, \psi_\alpha(\vec{x}\tau)\psi^\dagger_\beta(\vec{x}'\tau')\, e^{-S_E}}{\int \mathcal{D}(\psi^\dagger\psi)\, e^{-S_E}} \\ &= \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\, \underset{\text{time-ordering operator}}{T}\left[\hat{\psi}_\alpha(\vec{x}\tau)\hat{\psi}^\dagger_\beta(\vec{x}'\tau')\right]\right] \Big/ \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]\end{aligned}}$$

where the imaginary time dependence of the field operators is given by

$$\boxed{\begin{aligned}\hat{\psi}_\alpha(\vec{x}\tau) &= e^{(\hat{H}-\mu\hat{N})\tau}\, \hat{\psi}_\alpha(\vec{x})\, e^{-(\hat{H}-\mu\hat{N})\tau} \\ \hat{\psi}^\dagger_\beta(\vec{x}\tau) &= e^{(\hat{H}-\mu\hat{N})\tau}\, \hat{\psi}^\dagger_\beta(\vec{x})\, e^{-(\hat{H}-\mu\hat{N})\tau}\end{aligned}}$$

- Fetter and Walecka define

$$\boxed{\hat{K} \equiv \hat{H}-\mu\hat{N}}$$

which is like a grand canonical Hamiltonian, so the field operators are in a modified Heisenberg picture (F+W call them "Heisenberg" operators).

- For τ real, $\hat{\psi}(\vec{x}\tau)$ and $\hat{\psi}^\dagger(\vec{x},\tau)$ are not Hermitian adjoints, but they are if we continue to $\tau = it$.

- These definitions are useful because the grand canonical weighting operator $e^{-\beta(\hat{H}-\mu\hat{N})}$ is of the same form.

- The noninteracting version $\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')$ is found by replacing $\hat{H}$ by $\hat{H}_0$:

$$\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \frac{\mathrm{Tr}\, e^{-\beta(\hat{H}_0-\mu\hat{N})}\, T\left[\hat{\psi}_\alpha(\vec{x}\tau)\hat{\psi}^\dagger_\beta(\vec{x}'\tau')\right]}{\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}}}$$

(Recall $H_0 = \sum_{\vec{k}\lambda} \epsilon^0_k\, a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda}$.)

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- The noninteracting field operators are (Ω = volume here)

$$\boxed{\hat{\psi}(\vec{x}) = \frac{1}{\sqrt{\Omega}}\sum_{\vec{k}\lambda} e^{i\vec{k}\cdot\vec{x}}\,\eta_\lambda\, a_{\vec{k}\lambda}}$$

and

$$\boxed{\hat{\psi}^\dagger(\vec{x}) = \frac{1}{\sqrt{\Omega}}\sum_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{x}}\,\eta_\lambda^\dagger\, a_{\vec{k}\lambda}^\dagger}$$

and it is the $a_{\vec{k}\lambda}$ and $a^\dagger_{\vec{k}\lambda}$ operators that pick up the time dependence.

- It is straightforward to show that ($\hbar = 1$ everywhere)

$$\boxed{a_{\vec{k}\lambda}(\tau) = a_{\vec{k}\lambda} e^{-(\epsilon^0_k-\mu)\tau} \quad \text{and} \quad a^\dagger_{\vec{k}\lambda}(\tau) = a^\dagger_{\vec{k}\lambda} e^{(\epsilon^0_k-\mu)\tau}}$$

by evaluating commutators (exercise for the reader...)

- Let $\boxed{\langle a^\dagger_{\vec{k}\lambda} a_{\vec{k}'\lambda'}\rangle_0 \equiv \dfrac{\mathrm{Tr}\left[e^{-\beta(\hat{H}_0-\mu\hat{N})}\, a^\dagger_{\vec{k}\lambda} a_{\vec{k}'\lambda'}\right]}{\mathrm{Tr}\left[e^{-\beta(\hat{H}_0-\mu\hat{N})}\right]} = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}\, n^0_k}$

Why is it diagonal?
(This is an assumption that might be invalid for some systems)

with $\boxed{n^0_k = \dfrac{1}{e^{\beta(\epsilon^0_k-\mu)}+1}}$ our good old fermion occupation number.

- It follows that $\boxed{\langle a_{\vec{k}\lambda'} a^\dagger_{\vec{k}\lambda}\rangle_0 = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}(1-n^0_k)}$ [anticommutation relations]

- Now just plug and chug into $\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')$

$$\boxed{\begin{aligned}\tau>\tau' :\ & \frac{1}{\Omega}\sum_{\vec{k}\vec{k}'}\sum_{\lambda\lambda'} e^{i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')}(\eta_\lambda)_\alpha(\eta^\dagger_{\lambda'})_\beta\, e^{-(\epsilon^0_k-\mu)\tau+(\epsilon^0_{k'}-\mu)\tau'}\langle a_{\vec{k}\lambda} a^\dagger_{\vec{k}'\lambda'}\rangle_0 \\ &= \frac{1}{\Omega}\delta_{\alpha\beta}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-(\epsilon^0_k-\mu)(\tau-\tau')}(1-n_{k_0})\end{aligned}}$$

$$\boxed{\tau<\tau' :\ -\frac{1}{\Omega}\delta_{\alpha\beta}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-(\epsilon^0_k-\mu)(\tau-\tau')}(-n^0_k)}$$ ("$-$" from time ordering)

$\Rightarrow \mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')$ is the same as before!

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- Let's try the perturbative expansion of $\ln Z/Z_0$, which we relate to our thermodynamic observables by

intermediate notation not the volume here use $\tilde{\Omega}$

$$\boxed{\tilde{\Omega}(\Omega, T, \mu) - \tilde{\Omega}_0(\Omega, T, \mu) = -\frac{1}{\beta}(\ln Z - \ln Z_0) = -1/\beta \ln Z/Z_0}$$

- The replica method proof that $\ln Z/Z_0$ follows from keeping only the connected diagrams goes through just as in the model partition function case by considering $(Z/Z_0)^n$, which we construct simply by introducing duplicate Grassmann path integrals over ψ_i and $\bar{\psi}_i$ with $i = 1, \ldots, n$.

- To find out the precise Feynman rules, however, we'll need to carry out a couple orders of the expansion explicitly.

$$\Rightarrow \boxed{\frac{Z}{Z_0} = \frac{Z[\eta_\alpha, \eta^\dagger_\alpha]}{Z_0}\bigg|_{\eta=\eta^\dagger=0} = e^{\int_0^\beta d\tau \int d^3x \frac{\lambda}{2}\left(\frac{\delta}{\delta\eta_\alpha(x)}\frac{\delta}{\delta\eta_\beta(x)}\frac{-\delta}{\delta\eta^\dagger_\beta(x)}\frac{-\delta}{\delta\eta^\dagger_\alpha(x)}\right)} e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta} \eta_\delta}\bigg|_{\eta=\eta^\dagger=0}}$$

where the integral in the second exponential is a shorthand for

$$\boxed{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta} \eta_\delta \rightarrow \int_0^\beta d\tau_1 \int d^3x_1 \int_0^\beta d\tau_2 \int d^3x_2\, \eta^\dagger_\gamma(x_1)\, \mathcal{G}^0_{\gamma\delta}(\vec{x}_1\tau, \vec{x}_2\tau)\, \eta_\delta(x_2)}$$

- The $\vec{x}$ and τ dependence is actually rather easy to follow, so we will use the schematic form and just trace the spin indices.

- The other expansion we'll want to do is the Green's function $(\mathcal{G}, \mathcal{V}\mathcal{T})$, which we can write as

$$\boxed{\mathcal{G}_{\alpha\beta}(\vec{x}\tau; \vec{x}'\tau') = \frac{\frac{-\delta}{\delta\eta^\dagger_\alpha(\vec{x}\tau)}\frac{\delta}{\delta\eta_\beta(\vec{x}'\tau')}\, e^{-\frac{\lambda}{2}\int\left(\frac{\delta}{\delta\eta}\frac{\delta}{\delta\eta^\dagger}\right)^2} e^{\int \eta^\dagger \mathcal{G}^0 \eta}\Big|_{\eta=\eta^\dagger=0}}{e^{-\frac{\lambda}{2}\int\left(\frac{\delta}{\delta\eta}\frac{\delta}{\delta\eta^\dagger}\right)^2} e^{\int \eta^\dagger \mathcal{G}^0 \eta}\Big|_{\eta=\eta^\dagger=0}}}$$

and only connected diagrams survive.

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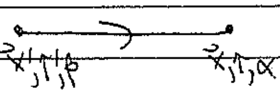
Let's do the leading order of $\mathcal{G}_{\alpha\beta}$ first:

$$\boxed{\begin{aligned}\mathcal{G}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') &= -\frac{\delta}{\delta\eta^{\dagger}_{\alpha}(\vec{x},\tau)}\frac{\delta}{\delta\eta_{\beta}(\vec{x}'\tau')}\left(1+\int\eta^{\dagger}_{\gamma}\mathcal{G}^{0}_{\gamma\delta}\eta_{\delta}+\dots\right)+O(\lambda)\\ &= +\mathcal{G}^{0}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') + O(\lambda)\end{aligned}}$$

- notice how the minus sign was eliminated after anticommuting $\frac{\delta}{\delta\eta}$ through $\eta^{\dagger}$.
- We didn't bother putting in the space and time coordinates in the expansion, since they just get set equal to those in the functional derivatives.
- Note that there is no $\frac{1}{2}$ in $\int\eta^{\dagger}\mathcal{G}^{0}\eta$, in contrast to the QM case where we had $\int jA^{-1}j$ with the same j's.
- Since we can tell the $\vec{x}$ end from the $\vec{x}'$ end, the Feynman rule will be to assign $\mathcal{G}^{0}_{\alpha\beta}$ to a line with an arrow:

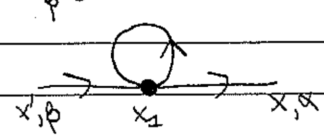


$\Rightarrow \mathcal{G}^{0}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') \equiv \delta_{\alpha\beta}\mathcal{G}^{0}(\vec{x}\tau;\vec{x}'\tau')$

* • [NOTE: Different conventions are used by different authors, which lead to minus signs (or worse) differing in the rules. The final answers for observables, of course, should be the same.]

- When we go to the next order, λ^{1}, there is only one connected diagram: (keeping only the terms surviving $\eta,\eta^{\dagger}\to 0$ at the end)

$$-\frac{\delta}{\delta\eta^{\dagger}_{\alpha}(x)}\frac{\delta}{\delta\eta_{\beta}(x')}\left(-\frac{\lambda}{2}\int d^{4}x_{1}\left(\frac{\delta}{\delta\eta_{\alpha'}}\frac{\delta}{\delta\eta_{\beta'}}\frac{\delta}{\delta\eta^{\dagger}_{\beta'}}\frac{\delta}{\delta\eta^{\dagger}_{\alpha'}}\right)\right)\frac{1}{3!}\left(\int\eta^{\dagger}_{\alpha_1}\mathcal{G}^{0}_{\alpha_1\beta_1}\eta_{\beta_1}\right)\left(\int\eta^{\dagger}_{\alpha_2}\mathcal{G}^{0}_{\alpha_2\beta_2}\eta_{\beta_2}\right)\left(\int\eta^{\dagger}_{\alpha_3}\mathcal{G}^{0}_{\alpha_3\beta_3}\eta_{\beta_3}\right)$$

= 

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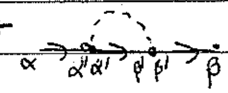
\+ 

- The second set of diagrams is just a visual aid to help with the spin algebra. ⇒ the two ends of •- - -• are the same space-time point x_1.
- Evaluate it after we establish Feynman rules...

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Now do the $\mathcal{O}(\lambda)$ part of Z/Z_0 (or $\ln Z/Z_0$; it is the same to that order sin $\ln(1+\epsilon) = \epsilon$ for ϵ small).

$$\ln\frac{Z}{Z_0} = \left(-\frac{\lambda}{2}\int d^4x \frac{\delta}{\delta\eta_\alpha}\frac{\delta}{\delta\eta_\beta}\frac{\delta}{\delta\eta^\dagger_\beta}\frac{\delta}{\delta\eta^\dagger_\alpha}\right)\left(\frac{1}{2!}\int dx_1\, \eta^\dagger_{\gamma_1} \mathcal{G}^0_{\gamma_1\delta_1}\eta_{\delta_1}\int dx_2\, \eta^\dagger_{\gamma_2}\mathcal{G}^0_{\gamma_2\delta_2}\eta_{\delta_2}\right) \quad ①$$

or ②

are the two distinct ways to match up the functional derivatives and the corresponding Grassmann variables.

- The net sign change from anticommuting is given by $(-1)^{\#\text{ of line crossings}}$ $\Rightarrow$ + for the first and − for the second.
- The first set is the "direct" term, while the second set is the "exchange" term.
- There are two ways of doing each type of term $\Rightarrow$ kills one $1/2$.

$$\Rightarrow\ \boxed{\begin{aligned} ① &= 2\times\left(-\frac{\lambda}{2}\right)\frac{1}{2!}\,\delta_{\alpha\gamma_1}\delta_{\alpha\delta_1}\delta_{\beta\gamma_2}\delta_{\beta\delta_2}\int d^4x\, \mathcal{G}^0_{\gamma_1\delta_1}(x,x^+)\,\mathcal{G}^0_{\gamma_2\delta_2}(x,x^+) \\ &= -\frac{\lambda}{2}\underbrace{\left(\delta_{\alpha\gamma_1}\delta_{\gamma_1\delta_1}\delta_{\delta_1\alpha}\right)}_{\delta_{\alpha\alpha}=\nu}\underbrace{\left(\delta_{\beta\gamma_2}\delta_{\gamma_2\delta_2}\delta_{\delta_2\beta}\right)}_{\delta_{\beta\beta}=\nu}\left(\int_0^\beta d\tau\int d^3x\right)\mathcal{G}^0(0,0^+)\,\mathcal{G}^0(0,0^+)\end{aligned}}$$

$$\boxed{\begin{aligned} ② &= -2\times\left(-\frac{\lambda}{2}\right)\frac{1}{2!}\,\delta_{\alpha\delta_1}\delta_{\alpha\gamma_2}\delta_{\beta\delta_2}\delta_{\beta\gamma_1}\int d^4x\, \mathcal{G}^0_{\gamma_1\delta_1}(x,x^+)\,\mathcal{G}^0_{\gamma_2\delta_2}(x,x^+) \\ &= +\frac{\lambda}{2}\underbrace{\left(\delta_{\alpha\delta_1}\delta_{\delta_1\gamma_1}\delta_{\gamma_1\beta}\delta_{\beta\delta_2}\delta_{\delta_2\gamma_2}\delta_{\gamma_2\alpha}\right)}_{\delta_{\alpha\alpha}=\nu}\left(\int_0^\beta d\tau\int d^3x\right)\mathcal{G}^0(0,0^+)\,\mathcal{G}^0(0,0^+)\end{aligned}}$$

- So one spin sums gives ν^2 (direct) and the other gives ν (exchange) and there is a minus sign between them. (We've used $\mathcal{G}^0_{\alpha\beta}\propto\delta_{\alpha\beta}$.)
- The vertex gives $-\lambda$ and the two is cancelled.
- We've written $\mathcal{G}^0(0,0^+)$ as an alternative to using the η infinitesimal in $\mathcal{G}^0$ to decide what to do at equal time.

$\vec{x}=\vec{x}'$ but $\tau'>\tau$ or $\tau'-\tau+\eta\Big|_{\tau-\tau'=0}$ $<\eta>0$ fixes which $\mathcal{G}$ function to take.

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- From page (A3), $\mathcal{G}^0(0;0^+) = \int \frac{d^3k}{(2\pi)^3}(-n_k^0)$
- Since there are no remaining dependencies of x or τ, $\int_0^\beta d\tau \int d^3x = \beta\Omega$

$$\Rightarrow \boxed{\ln\frac{Z}{Z_0} = (\beta\Omega)\left(+\frac{\lambda}{2}\right)(\nu^2-\nu)\left(\int\frac{d^3k}{(2\pi)^3}n_k^0\right)^2 + O(\lambda^2) = +\beta(\tilde{\Omega}-\tilde{\Omega}_0)}$$

or

$$\boxed{\tilde{\Omega} = \tilde{\Omega}_0 + \Omega\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\left(\nu\int\frac{d^3k}{(2\pi)^3}n_k^0\right)^2 + O(\lambda^2)}$$

- but $\nu\int\frac{d^3k}{(2\pi)^3}n_k^0$ $\left(\text{or } \frac{1}{\Omega}\sum_{k,\alpha} n_k^0\right)$ is just the density (the thermal averaged density).
- There is no μ dependence in the $O(\lambda)$ term, so in the $T=0$ limit,

$$\boxed{N = \frac{\partial\Omega}{\partial\mu} = \frac{\partial\Omega_0}{\partial\mu}} \quad \text{and} \quad \boxed{\tilde{\Omega}_0 = E_0 + \mu N}$$

$$\overset{T=0}{\Rightarrow} E = E_0 + E_1 = \frac{3}{5}\frac{\hbar^2k_f^2}{2m}\cdot N + V\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho^2$$

or $$\boxed{\frac{E^{(0)}}{N} + \frac{E^{(1)}}{N} = \frac{3}{5}\frac{\hbar^2k_f^2}{2m} + \frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho}$$

- This result agrees with a simple perturbative calculation using 2nd quantization.

- Note that we got the correct answer immediately for the exchange part without any change of variables or tricky integration regions.
  - The calculation would be considerably trickier if we didn't have a delta function interaction (ie, if it had finite or infinite range) but the generalization of our procedure is relatively easy.

- Note also that we rather trivially ended up in momentum space because of the $\mathcal{G}^0(0,0^+)$'s.

- Generalizing from $\frac{-\nabla^2}{2m}$ to $\frac{-\nabla^2}{2m}+U(\vec{x})$, with a background field $U(\vec{x})$ is simple (see later),

(A11)

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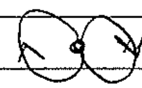
Ok, so what about the Feynman diagram and rules?

· One vertex, two lines 

$\Rightarrow$ vertex $\bullet$ $-\lambda$

· The "$-$" comes from the minus sign in front of the action in the exponent of the path integral.

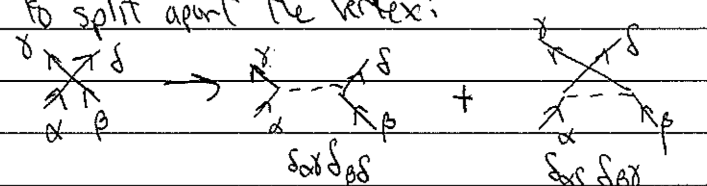
· The cancellation of the $\frac{1}{2}$ in front of the λ is like the 4! factor in our model. We get 2 instead of 4! because the two ends of each line (noninteracting Green's function) are different and we keep direct and exchange separate.

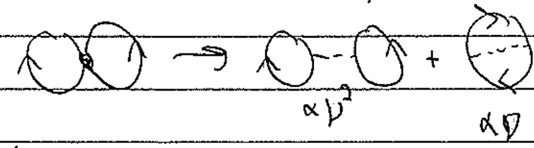
· Each line gets $\mathcal{G}^0_{\alpha\beta}(x,x')$ with x and x' determined by the vertices (or external points) it connects.

· Each vertex gets a space-time point x_i and we integrate $\int d^4x_i$ at the end.

· The vertices have two incoming lines and two outgoing lines.

· The spin sums follow the fermion lines around, until they close on themselves, yielding a net factor of ν. At each vertex there are two choices of directions to go. One way to follow the spin is to split apart the vertex:



so the diagrams are 

There is a relative sign, which we can account for in different ways. One convenient way is to just do the spin sums and substitute $-\nu$ for every $\delta_{\alpha\alpha}$ factor.

Note: If there are spin dependent interactions, one simply inserts the appropriate spin-matrices at the vertices.

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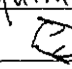
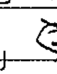
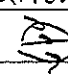
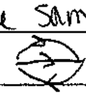
Finally, to get the overall factor correct, we need a symmetry factor. These diagrams have lines with arrows so the rules are modified to:

1. no factor anymore ⇒ 1 always
2. equivalent lines must have arrows in the same direction:  $\Rightarrow 1$,  $\frac{1}{2!}$,  $\frac{1}{3!}$,  $\frac{1}{2!}$ and on on
3. permutations must transform the diagram into itself, including the arrows.

To summarize, the rules for the n^{th} order contribution to $\ln(Z/Z_0)$ at temperature $T = 1/\beta$:

a. Draw all distinct, full connected diagrams with n vertices. Distinct diagrams are those that cannot be deformed to coincide with each other, including the directions of arrows.

b. Assign a spacetime point $x_i = (\vec{x}_i, \tau_i)$ to each vertex and a factor $(-\lambda)$. Each internal line gets a factor $\mathcal{G}^0_{\alpha\beta}(x_1, x_2)$ running from x_2 to x_1. The vertex lines each have a spin index. For spin-independent interactions the two-body vertices have the structure $(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ where α, β are incoming spins and γ, δ are outgoing spins.

c. Do the spin summations and substitute -2 for each $\delta_{\alpha\alpha}$ in a closed fermion loop.

d. Integrate $\int dx_i$ over all x_i. (We'll discuss how to deal with divergences later.)

e. Multiply by a symmetry factor as indicated above.

Pretty easy, huh?

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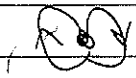
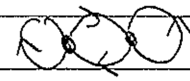
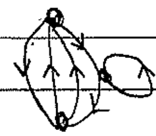
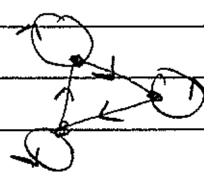
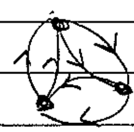
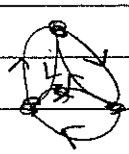
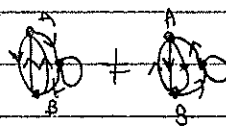
Diagram topology and symmetry factor practice:

λ^1:  ② $1/2$ ③ 1 $\Rightarrow \frac{1}{2}$

λ^2:  ② $\frac{1}{2}\cdot\frac{1}{2}$ ③ $\frac{1}{2} \Rightarrow \frac{1}{8}$  ② 1 ③ $\frac{1}{2} \Rightarrow \frac{1}{2}$

	②	③	total
λ^3: 	1	$\frac{1}{2}$	$\frac{1}{2}$
	$\frac{1}{2}$	1	$\frac{1}{2}$
	1	$\frac{1}{3}$	$\frac{1}{3}$
	$(\frac{1}{2})^3$	$\frac{1}{3}$	$\frac{1}{24}$
	1	$\frac{1}{3!}$	$\frac{1}{6}$

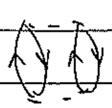
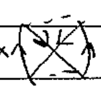
bubble identifies the way from A to B

~~friday~~: 

add arrows and factors

Try some spin sums. What are the factors with the λ^2 diagrams?

Use Mathematica notebook.

 $\Rightarrow 2\times$  $2\times$ 

$\delta_{\delta\delta'}\delta_{\alpha\gamma}\delta_{\gamma'\beta}\left(\delta_{\delta\delta'}\delta_{\gamma'\gamma} + \delta_{\delta\gamma}\delta_{\gamma'\delta}\right)$

$-\nu\delta_{\alpha\beta} + \delta_{\alpha\beta} \Rightarrow (-\nu+1)\delta_{\alpha\beta}$

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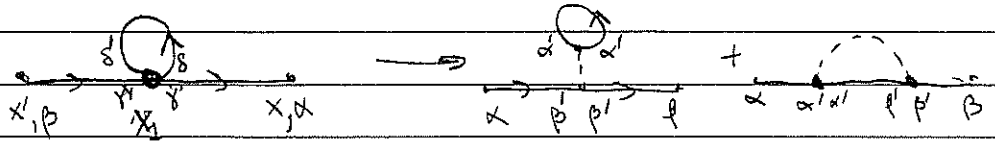
- If we now consider calculating the single-particle Green's function (...):



$\Rightarrow(-\nu)$ $\qquad\qquad$ $\Rightarrow(+1)$

- The spin sums are indicated by the exploded diagrams.
  - In both cases we end up the $\delta_{\alpha\beta}$ connecting the outside lines, the outside lines but the direct term has an additional spin sum in the tadpole $\Rightarrow$ factor of $(-\nu)$ by our Feynman rules.

- The rest of the diagram is evaluated trivially: (Feynman rules add external points):

$$\boxed{\mathcal{G}^{(1)}_{\alpha\beta}(\vec{x}\tau, \vec{x}'\tau') = -\int d^4x_1\, \mathcal{G}^0_{\alpha\gamma}(x,x_1)\left[\delta_{\gamma\delta}(-\lambda)(-\nu+1)\mathcal{G}^0(x_1,x_1^+)\right]\mathcal{G}^0_{\delta\beta}(x_1,x')}$$

- As before, $\boxed{\mathcal{G}^0(x_1,x_1^+) = \mathcal{G}^0(0,0^+) = \int\frac{d^3k}{(2\pi)^3}(-n_k^0) = -\frac{\rho}{\nu}}$ for a uniform system.

- The structure here is of the form $\mathcal{G}^0(-\Sigma)\mathcal{G}^0$, where we suppress the integrations and spin indices. Any contribution to $\mathcal{G}$ at any order will always have $\mathcal{G}^0$'s bracketing a piece in the middle, which we call the "self-energy." (It is conventionally defined with a minus sign at $T\neq 0$).
  - In the present case Σ is a constant and diagonal in spin. More generally it is a function of two space-time points and is spin dependent.
- We'll talk much more about the self-energy and the "proper" self energy below.

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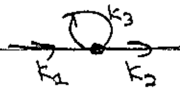
If we substitute

$$\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \delta_{\alpha\beta}\frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-(\epsilon_k-\mu)(\tau-\tau')} \times\left[\Theta(\tau-\tau')(1-n^0_k)-\Theta(\tau'-\tau)n^0_k\right]}$$

into the equation for $\mathcal{G}^{(1)}_{\alpha\beta}$, we initially have to sum over a different variable $\vec{k}_1, \vec{k}_2, \ldots$ for each propagator line.



- But all of the $\vec{x}_1$ dependence will then be explicit:

$$\boxed{\int d^3x_1\, e^{-i\vec{k}_2\cdot\vec{x}_1}\, e^{i\vec{k}_1\cdot\vec{x}_1} = \int d^3x_1\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{x}_1} = (2\pi)^3\delta^3(\vec{k}_1-\vec{k}_2).}$$

$\Rightarrow$ momentum conservation.

- we also have a $e^{i\vec{k}_3\cdot(\vec{x}_1-\vec{x}_1)} = 1$ factor, which just says that $\vec{k}_3$ is both entering and exiting the vertex, so it doesn't add any new constraint.

- This pattern will repeat with any diagram, which means we can eliminate all $\vec{x}$ integrations from the start and replace them directly with momentum sums (which become integrals as $V\to\infty$) in the propagators. The Feynman rules will include the prescription to conserve momentum at the vertices.

- To deal with the time dependence we will end up doing something similar and switch to frequency space.
  - We have to be a bit more careful, since the time interval is from 0 to β and we have periodic (or antiperiodic in this case) boundary conditions.
  - We'll come back to this.

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Let's do a bit of a recap...

Summary points:

- We can calculate $\ln Z/Z_0$ or $\mathcal{G}_{\alpha\beta}$ by diagrams.
  - We've used contact interactions and uniform systems, but generalizations follow the same pattern.
- Recall

$n^0_k = \frac{1}{1+e^{\beta(\epsilon_k-\mu)}}$

$$\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-(\epsilon^0_k-\mu)(\tau-\tau')} \times\left[\theta(\tau-\tau'-\eta)(1-n^0_k)-\theta(\tau'-\tau+\eta)n^0_k\right]}$$

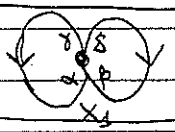
$$\underset{\vec{x}'\tau'}{\beta}\longrightarrow\underset{\vec{x}\tau}{\alpha}$$

- Let's do the first orders of $\ln Z/Z_0$ in the $T\to 0$ limit
  - We need the noninteracting part. Do that on the next couple of pages.
  - We already have the order λ part from the notes, but let's do it from the Feynman rules:

Order λ^1

a. Draw all distinct diagrams with 1 vertex 

b. Assign $x_1 = (\vec{x}_1,\tau_1)$ to the vertex and $-\lambda$.

$\mathcal{G}^0_{\alpha\gamma}(x_1,x_1)$  $\mathcal{G}^0_{\beta\delta}(x_1,x_1)$ vertex $(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})$

- look at spin sum Mathematica notebook

c. Spins $\underbrace{\delta_{\alpha\gamma}\delta_{\beta\delta}}_{\text{from } \mathcal{G}^0\text{'s}}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) = \delta_{\alpha\alpha}\delta_{\beta\beta}+\delta_{\alpha\delta}\delta_{\delta\gamma} = (-\nu)^2+\delta_{\gamma\gamma}$
$= \nu^2-\nu = \nu(\nu-1)$

d. $\boxed{\mathcal{G}^0(x_1,x_1) = \int\frac{d^3k}{(2\pi)^3}(-n^0_k) = \mathcal{G}^0(0,0)}$ $\quad$ $\boxed{\int dx_1\,[\mathcal{G}^0(0,0)]^2 = [\mathcal{G}^0(0,0)]^2\int_0^\beta d\tau\int d^3x_1 = \beta V[\mathcal{G}^0(0,0)]^2}$

e. symmetry factor $1\cdot\frac{1}{2}\cdot 1 \Rightarrow \frac{1}{2}$

$$\Rightarrow \ln\frac{Z}{Z_0} = \frac{1}{2}(\beta V)(-\lambda)(\nu^2-\nu)\left(\int\frac{d^3k}{(2\pi)^3}n^0_k\right)^2 = -\beta(\tilde{\Omega}-\tilde{\Omega}_0)$$

or $\tilde{\Omega} = \tilde{\Omega}_0 + V\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\left(\nu\int\frac{d^3k}{(2\pi)^3}n^0_k\right)^2$

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- Note that Z_0 is just the partition function for a non-interacting system of fermions.
  - We could calculate it from the path integral — our formula for Gaussian integrals tells us it is proportional to the determinant of $(\mathcal{G}_0)^{-1}$ $\Rightarrow$ later
  - However, we could just as well take it from a 2nd quantization calculation. In particular, on the following pages are derived results for $\ln Z_0$ (which is what we need, really) for a noninteracting Fermi gas with degeneracy ν in a box of volume Ω.

- Let's summarize some of the basic results:

$$\boxed{\begin{aligned}\tilde{\Omega}_0(V,T,\mu) &= -\frac{1}{\beta}\sum_{k\alpha}\ln\left(1+e^{-\beta(\epsilon_k-\mu)}\right)\\ &\xrightarrow[\Omega\to\infty]{} -\frac{g}{\beta}\frac{\Omega}{(2\pi)^3}\int d^3k\,\ln\left(1+e^{-\beta(\epsilon_k-\mu)}\right)\end{aligned}}$$

where $\boxed{\epsilon_k = \frac{\hbar^2k^2}{2m}}$

A change of variables to ϵ_k (which we just called ϵ) yields forms that are simpler to evaluate at finite temperature:

$$\Rightarrow \boxed{\begin{aligned}\tilde{\Omega}_0 &= -\frac{\nu}{\beta}\frac{\Omega}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\,\epsilon^{1/2}\ln\left(1+e^{\beta(\mu-\epsilon)}\right)\\ \underset{\text{integration}}{\overset{\text{partial}}{=}}& \frac{\nu\Omega}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\int_0^\infty d\epsilon\,\frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1} = P\Omega\end{aligned}}$$

We can find N from thermodynamics:

$$\boxed{\begin{aligned}N = -\frac{\partial\tilde{\Omega}_0}{\partial\mu} &= -\frac{\nu\Omega}{\beta(2\pi)^3}\int d^3k\,\frac{1}{1+e^{-\beta(\epsilon_k-\mu)}}e^{-\beta(\epsilon_k-\mu)}\beta = \frac{\nu\Omega}{(2\pi)^3}\int d^3k\,\underbrace{\frac{1}{e^{\beta(\epsilon_k-\mu)}+1}}_{n_k^0}\\ &= \frac{\nu\Omega}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\,\frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1}\end{aligned}}$$

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We'll be taking the zero temperature limit in the interacting case; let's warm up with the noninteracting case.

- We'll calculate the noninteracting ground-state energy E_0 from

$$\boxed{E_0 = \lim_{T\to 0}\left(\tilde{\Omega}_0 + TS + \mu N\right) = \lim_{\beta\to\infty}\left(\tilde{\Omega}_0 + \mu_0 N\right)}$$

- Start with N. We'll use the expression with the integral over k.

$$n_k^0 \equiv \boxed{\frac{1}{e^{\beta(\epsilon_k-\mu)}+1} \longrightarrow \begin{cases} 0 & \text{if } \epsilon_k-\mu_0 > 0 \\ 1 & \text{if } \epsilon_k-\mu_0 < 0 \end{cases}}$$

$\Rightarrow$ the occupation number $n_k \to \theta(\mu_0 - \epsilon_k)$.

So we fill levels until the last filled state has energy μ. Since this is the Fermi energy in the non-interacting Fermi gas, we have the result:

$$\text{at } T=0 \quad \boxed{\mu_0 = \epsilon_F = \frac{\hbar^2 k_F^2}{2m}} \Rightarrow \boxed{\rho = \frac{\nu}{6\pi^2}\left(\frac{2m\mu_0}{\hbar^2}\right)^{3/2}} = \frac{\nu}{6\pi^2}k_F^3$$

If we now consider $\tilde{\Omega}_0$, then let $z \equiv \beta(\epsilon_k - \mu_0)$.

If $z \to +\infty$, then $\ln(1+e^{-z}) \to 0$

If $z \to -\infty$, then $\ln(1+e^{-z}) \to -z$

$$\text{So} \quad \boxed{\begin{aligned} \tilde{\Omega}_0 \xrightarrow[\beta\to\infty]{} & -\frac{\nu}{\beta}\frac{\Omega}{(2\pi)^3}\int d^3k \, -\beta(\epsilon_k-\mu_0)\,\theta(\mu_0-\epsilon_k) \\ &= +\frac{\nu\Omega}{(2\pi)^3}\int d^3k\,\epsilon_k\,\theta(\mu_0-\epsilon_k) - \mu_0\frac{\nu\Omega}{(2\pi)^3}\int d^3k\,\theta(\mu_0-\epsilon_k) \\ &= \sum_{k,\alpha}^{k_F}\frac{\hbar^2k^2}{2m} - \mu_0\sum_{k,\alpha}^{k_F} 1 = \frac{3}{5}\epsilon_F\Omega - \mu_0 N \end{aligned}}$$

From which we recover $\boxed{E_0 = \frac{3}{5}\epsilon_F\rho\Omega = \frac{3}{5}\epsilon_F N}$

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As before, the factor $\boxed{\nu \int \frac{d^3k}{(2\pi)^3} n_k^0 = \frac{1}{\Omega}\sum_{k,\alpha} n_k^0}$ is just the (thermal averaged) density of the system.

• The $O(\lambda)$ term has no μ dependence, so $\frac{\partial \tilde\Omega_1}{\partial\mu}=0$. In the $T=0$ limit this means

$$\boxed{N = \frac{\partial\tilde\Omega}{\partial\mu} = \frac{\partial\tilde\Omega_0}{\partial\mu}} \quad \text{and} \quad \boxed{\tilde\Omega_0 = E_0 + \mu N}$$

$$\Rightarrow \boxed{E = E_0 + E_1 = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\cdot N + \Omega\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho^2}$$

or

$$\boxed{\frac{E^{(0)}}{N}+\frac{E^{(1)}}{N} = \frac{3}{5}\frac{\hbar^2k_F^2}{2m} + \frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho = \frac{3}{5}\frac{\hbar^2k_F^2}{2m}+\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\left(\frac{\nu k_F^3}{6\pi^2}\right)}$$

energy per particle

• shortly we'll find that $\lambda = \frac{4\pi a_s}{m}$ where a_s is the "scattering length"

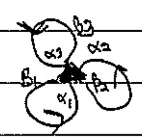
Energy density is $\frac{E}{V} = \frac{E}{N}\cdot\frac{N}{V} = \frac{E}{N}\rho$

$$\Rightarrow \boxed{\frac{E}{N} = \frac{k_F^2}{2m}\left[\frac{3}{5} + \underbrace{(\nu-1)\frac{2}{3\pi}}_{\text{order unity}}(k_F a_s) + O((k_F a_s)^2)\right]}$$

So looks like an expansion in $k_F a_s$ ⇒ return to this below.

• Note that if $\nu=1$, we get that the energy per particle is just that of the non interacting system.

• What if we had a 3-body interaction, $\propto \psi^\dagger_\gamma \psi^\dagger_\beta \psi^\dagger_\alpha \psi_\alpha \psi_\beta \psi_\gamma$

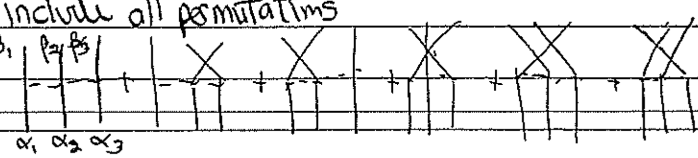


$\Rightarrow \propto \beta V$ from $\int d^3x_1$

$\propto \rho^3$ from $[\mathcal{G}^0(0,0)]^3$

What about spin sum? Generalize the 2-body vertex rule

⇒ include all permutations



+ signs because "$-\nu$" rule takes care of "$-$"'s.

$$\Rightarrow \underbrace{\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}}_{\text{from propagator}}\Big(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3} + \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_3}$$
$$+ \delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_1} + \delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}\Big)$$

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If we give this to the Mathematica notebook that uses the deltasimplify package, we obtain $-(\nu-2)(\nu-1)\nu$

- Why does this make sense?
  - $\Rightarrow$ now we need at least 3 different "spins".

- Comments on the deltasimplify.m package
  - A Mathematica package has definitions that can be loaded for use in a Mathematica notebook.
  - The "usage" commands give the help text returned when you do ?DeltaSimplify or ?del.
  - A Private context is used — this is like a separate namespace so there isn't interference if the user happens to use the same variable names.
    - Note that nu is in the global context.
  - The deltasimprules are a series of pattern matching rules: if the left side is matched, replace it with the right side.
    - The first rule ensures that all the products of sums of δ's are expanded out, so that we have $\delta_{ab}\delta_{bc}$ etc. multiplying each other.
    - Note that del[a,b] is just an object with two slots that has some rules associated with it!
    - $\delta_{ab}\delta_{bc} = \delta_{ac}$ from the first, and then all permutations.
    - finally, the key rule: $\delta_{aa} = -\nu$.
  - The DeltaSimplify command, when applied to an expression, simply applies the rules over and over until the expression stops changing.
    - don't worry about the obscure syntax. Just copy it for your own rules!

11/21/14

What would a spin-dependent force look like?

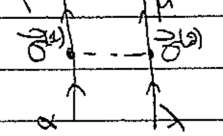
· If short-ranged:

$$\boxed{V_s(\vec{x}_1,\vec{x}_2)_{\alpha\beta,\lambda\mu} = \lambda_s\, \vec{\sigma}(1)_{\alpha\beta}\cdot\vec{\sigma}(2)_{\lambda\mu}\, \delta^3(\vec{x}_1-\vec{x}_2)}$$

where the Pauli matrices are labeled for particle 1 or particle 2.

· So the difference from the spin-independent case is

$$\delta^{(1)}_{\alpha\beta} \to \vec{\sigma}^{(1)}_{\alpha\beta} \quad \text{and} \quad \delta_{\lambda\mu}(2) \to \vec{\sigma}^{(2)}_{\lambda\mu}$$



so the σ-matrices live along the lines for particle 1 and 2

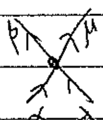
· The interaction term in the Lagrangian is generally

$$\tfrac{1}{2}\psi^\dagger_\alpha(x_1)\psi^\dagger_\lambda(x_2)V(\vec{x}_1,\vec{x}_2)_{\alpha\beta,\lambda\mu}\psi_\mu(x_2)\psi_\beta(x_1)$$

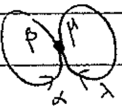
spin-independent $V(\vec{x}_1,\vec{x}_2)_{\alpha\beta,\lambda\mu} = \lambda\,\delta_{\alpha\beta}\delta_{\lambda\mu}\delta^3(\vec{x}_1-\vec{x}_2)$

$$\Rightarrow \tfrac{\lambda}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\lambda(x)\psi_\lambda(x)\psi_\alpha(x) \quad \text{as before.}$$

Now spin-dependent $\Rightarrow \tfrac{\lambda_s}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\lambda(x)\psi_\mu(x)\psi_\beta(x)\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\lambda\mu}$

· The Feynman rule  $\Rightarrow (-\lambda)\,\delta_{\alpha\beta}\delta_{\lambda\mu} + \delta_{\alpha\mu}\delta_{\lambda\beta}$ and $-\nu$ for $\delta_{\alpha\alpha}$

now becomes (replacing δ's with σs): $(-\lambda_s)(\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\lambda\mu} + \vec{\sigma}_{\alpha\mu}\cdot\vec{\sigma}_{\lambda\beta})$

· Try the bowtie  $(-\lambda_s)\tfrac{1}{2}[\mathcal{G}^0(0,0)]^2\beta V \times \delta_{\alpha\beta}\delta_{\lambda\mu}(\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\lambda\mu} + \vec{\sigma}_{\alpha\mu}\cdot\vec{\sigma}_{\lambda\beta})$

Spin factor is $\cancel{\vec{\sigma}_{\alpha\alpha}\cdot\vec{\sigma}_{\lambda\lambda}}^{\,0} + \vec{\sigma}_{\alpha\mu}\cdot\vec{\sigma}_{\mu\alpha} = 3\,\delta_{\alpha\alpha} = -3\nu$ instead of $\nu(\nu-1)$

from dot product

$\Rightarrow$ we get 3 times the Fock term from the spin-independent case.

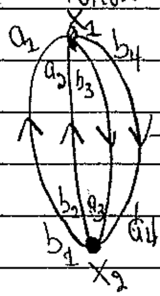
· The average over spin makes the Hartree term vanish.

11/1/14

Let's do the famous beachball diagram... (With the $T\to 0$ or $\beta\to\infty$ limit in mind)

· Follow the Feynman rules for (using $\alpha\to a$, $\beta\to b$)

Convention to label lines pointing from b to a $\Rightarrow$



· symmetry factor $1\times\left(\frac{1}{2}\right)^2\times\frac{1}{2}=\frac{1}{8}$

$$\left[\ln\frac{Z}{Z_0}\right]_{\text{beachball}} = -\beta\Omega^{(2)}_{\text{beachball}}$$

$$= \frac{1}{8}(-\lambda)^2\int d\tau_1 d^3x_1\int d\tau_2 d^3x_2\, \mathcal{G}^0_{k_1}(x_1,x_2)\,\mathcal{G}^0_{k_2}(x_1,x_2)\,\mathcal{G}^0_{k_3}(x_2,x_1)\,\mathcal{G}^0_{k_4}(x_2,x_1)$$

Mathematica says $2\nu(\nu-1)$ $\Rightarrow$

$$\times\,\underbrace{\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_4}}_{\text{from } \mathcal{G}^0\text{'s}}\underbrace{\left(\delta_{a_1b_3}\delta_{a_2b_4}+\delta_{a_1b_4}\delta_{a_2b_3}\right)}_{\text{top vertex}}\underbrace{\left(\delta_{a_3b_1}\delta_{a_4b_2}+\delta_{a_3b_2}\delta_{a_4b_1}\right)}_{\text{bottom vertex}}$$

· We've peeled off the spin labels from the $\mathcal{G}^0$'s and added labels for the value of k in

$$\mathcal{G}^0_{k_1}(\vec{x}_1\tau_1,\vec{x}_2\tau_2) = \int\frac{d^3k_1}{(2\pi)^3}\, e^{i\vec{k}_1\cdot(\vec{x}_1-\vec{x}_2)}\, e^{-(\epsilon^0_{k_1}-\mu)(\tau_1-\tau_2)} \times\left[\Theta(\tau_1-\tau_2-\eta)(1-n^0_k)-\Theta(\tau_2-\tau_1+\eta)\,n^0_k\right]$$

to each line.

· All of the $\mathcal{G}^0$'s in the beachball depend on $\vec{x}_1-\vec{x}_2$, $\tau_1-\tau_2$ so we can do those integrals:

$$\int d^3x_1\int d^3x_2\, e^{i(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\cdot(\vec{x}_1-\vec{x}_2)} \underset{\vec{y}=\vec{x}_1-\vec{x}_2}{=} \int d^3x_1\int d^3y\, e^{i(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\cdot\vec{y}}$$

$$= V(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)$$

$\Rightarrow$ momentum conservation! (always happens $\Rightarrow$ momentum Feynman rules)

· Note that we get a single factor of the volume Ω. What if there were disconnected diagrams? E.g. $\Rightarrow \Omega^2$ (and so on)

· But $\ln Z/Z_0 \propto \Omega-\Omega_0$, which should be extensive.

$\Rightarrow$ the linked cluster expansion ensures extensivity!

(A23)

11/1/14

What about τ_1 and τ_2 integrals?

- Shift from $[0,\beta]$ to $[-\beta/2,\beta/2]$ so $\beta\to\infty$ is more convenient.

$$\boxed{\begin{aligned}\tau_1' &= \tau_1-\beta/2\\ \tau_2' &= \tau_2-\beta/2\end{aligned}} \qquad \boxed{\int_0^\beta d\tau_1\int_0^\beta d\tau_2 = \int_{-\beta/2}^{\beta/2}d\tau_1'\int_{-\beta/2}^{\beta/2}d\tau_2' \xrightarrow[\beta\to\infty]{} \beta\int_{-\infty}^{\infty}dy_0} \quad \text{where } y_0=\tau_1-\tau_2$$

What about θ functions? $\theta(\tau_1-\tau_2)\times\theta(\tau_1-\tau_2)=\theta(\tau_1-\tau_2)$

$$\theta(\tau_1-\tau_2)\times\theta(\tau_2-\tau_1)=0$$

$\Rightarrow$ only 2 terms survive and they are equal (just interchange $1,2\Leftrightarrow 3,4$)

so keep one of them and overall factor of 2

$$\Rightarrow \boxed{\begin{aligned}&2\beta\int_{-\infty}^{\infty}dy_0\,\theta(y_0)\,e^{-[(\epsilon^0_{k_1}-\mu)+(\epsilon^0_{k_2}-\mu)-(\epsilon^0_{k_3}-\mu)-(\epsilon^0_{k_4}-\mu)]y_0}\times(1-n^0_{k_1})(1-n^0_{k_2})(1-n^0_{k_3})(1-n^0_{k_4})\\ &=2\beta(1-n^0_{k_1})(1-n^0_{k_2})n^0_{k_3}n^0_{k_4}\int_0^\infty dy_0\,e^{-(\epsilon^0_{k_1}+\epsilon^0_{k_2}-\epsilon^0_{k_3}-\epsilon^0_{k_4})y_0}\\ &\qquad\qquad + \frac{1}{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4}\end{aligned}}$$

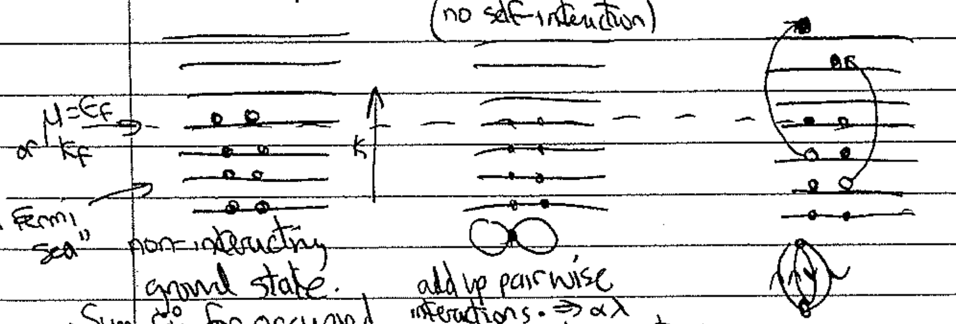
The μ's cancel! Only μ dependence in

This is an energy denominator!

Put in all together! symmetry factor, spin sum, 2 sets of θ's

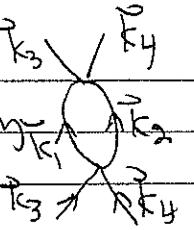
$$\boxed{\begin{aligned}\left(\ln\frac{Z}{Z_0}\right)_{\text{beachball}} &= -\beta\Omega^{(2)}_{\text{beachball}} = \frac{1}{8}[2\lambda(2\lambda-1)]\cdot 2\,\beta\Omega\int\frac{d^3k_1\cdots d^3k_4}{(2\pi)^{12}}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\frac{(-\lambda)^2}{\epsilon^0_{k_1}+\epsilon^0_{k_2}-\epsilon^0_{k_3}-\epsilon^0_{k_4}}\\ &= -\beta\Omega\,\mathcal{E}^{(2)}_{\text{beachball}} \leftarrow T=0 \text{ energy density}\end{aligned}}$$

Consider pictures in the $T\to 0$ ($\beta\to\infty$) limit



"2p-2h" excited states have 2 particles above, leaving two "holes" below

$\Rightarrow$ sum over allowed possibilities

11/1/14. Scattering:  $\Leftarrow$ virtual excitations (violate energy conservation)

Momentum conservation holds: $\vec{k}_1+\vec{k}_2=\vec{k}_3+\vec{k}_4$ is required (as vectors)

· Why not 1p-1h? (can't conserve momentum)

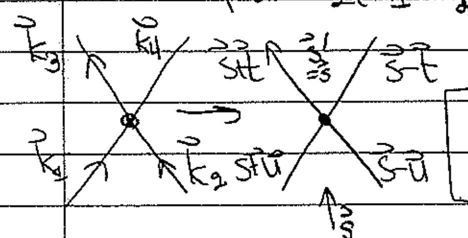
· This is 2nd-order perturbation theory for many-body system!

$$\boxed{H=H_0+H_1,\ H_0|\Phi_n\rangle=E_n|\Phi_n\rangle} \Rightarrow \boxed{\delta E^{(2)}=-\sum_{n\neq 0}\frac{\langle\Phi_0|H_1|\Phi_n\rangle\langle\Phi_n|H_1|\Phi_0\rangle}{E_n-E_0}}$$

(note order and "−" out front)

Switch variable to make clearer what is happening
$\Rightarrow$ identify total and relative momenta

e.g. $k_F\vec{s}=\frac{1}{2}(\vec{k}_1+\vec{k}_2)=\frac{1}{2}(\vec{k}_3+\vec{k}_4)\equiv k_F\vec{s}\,'$ momentum [could define without ½]
and $k_F\vec{u}=\frac{1}{2}(\vec{k}_1-\vec{k}_2)$, $k_F\vec{t}=\frac{1}{2}(\vec{k}_3-\vec{k}_4)$



Claim:
$$\boxed{\int d^3k_1\cdots d^3k_4\,\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)=8k_F^9\int d^3s\,d^3u\,d^3t}$$

[Proof: Use $k_F\vec{s}\,'$ $\Rightarrow \delta(k_{1x}+k_{2x}-k_{3x}-k_{4x})=\delta(2k_F(s_x-s_x'))=\frac{1}{2k_F}\delta(s_x-s_x')$
for each Cartesian coordinate. Then $k_{1x}=k_F(s_x+u_x)$ and so on

$$\Rightarrow\int dk_{1x}dk_{2x}dk_{3x}dk_{4x}\,\delta(k_{1x}+k_{2x}-k_{3x}-k_{4x})=k_F^4\int ds_x\,ds_x'\,du_x\,dt_x\,\frac{1}{2k_F}\delta(s_x-s_x')\left|\begin{matrix}\frac{\partial k_{1x}}{\partial s_x} & \frac{\partial k_{2x}}{\partial s_x}\\ \frac{\partial k_{1x}}{\partial u_x} & \frac{\partial k_{2x}}{\partial u_x}\end{matrix}\right|\left|\begin{matrix}\frac{\partial k_{3x}}{\partial s_x'} & \frac{\partial k_{4x}}{\partial s_x'}\\ \frac{\partial k_{3x}}{\partial t_x} & \frac{\partial k_{4x}}{\partial t_x}\end{matrix}\right|$$

$$=k_F^3(2\cdot 2\cdot\tfrac{1}{2})\int ds_x\,du_x\,dt_x$$

$$\left|\left|\begin{matrix}+1 & +1\\ -1 & -1\end{matrix}\right|\right| = 2$$

$\Rightarrow 2^3=8$ overall. QED]

· note: choosing different variables might make this easier if Jacobian could be simpler!

11/1/14

Now

$$\varepsilon^0_{k_1}+\varepsilon^0_{k_2}=\frac{k_f^2}{2m}\left((s+u)^2+(s-u)^2\right)=\frac{k_f^2}{m}(s^2+u^2)$$

$$-\quad \varepsilon^0_{k_3}+\varepsilon^0_{k_4}=\frac{k_f^2}{2m}\left((s+t)^2+(s-t)^2\right)=\frac{k_f^2}{m}(s^2+t^2)$$

$$\overline{\frac{k_f^2}{2m}(u^2-t^2)}$$

So finally, in the $T\to 0$ where $\boxed{n_k\to\theta(k_f-|\vec{k}|)}$, $\boxed{1-n_k\to\theta(|\vec{k}|-k_f)}$ we have

$$\boxed{\varepsilon^{(2)}_{bb}=-4\lambda^2 m\,\nu(\nu-1)\,k_f^7\int\frac{d^3s}{(2\pi)^3}\frac{d^3t}{(2\pi)^3}\frac{d^3u}{(2\pi)^3}\,\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\,\theta(|\vec{s}+\vec{u}|-1)\,\theta(|\vec{s}-\vec{u}|-1)\times\frac{1}{u^2-t^2}}$$

can this vanish?

- Now s and t integrals are limited so $|\vec{s}|\le 1$, $|\vec{t}|\le 1$ but look at $\vec{u}$ large

$$\Rightarrow\ \theta(|\vec{s}+\vec{u}|-1)=\theta(|\vec{s}-\vec{u}|-1)\longrightarrow 1$$

$$\Rightarrow\ \int_{u\ \text{large}}\frac{d^3u}{(2\pi)^3}\frac{1}{u^2-t^2}\ \propto\int^{\infty}\frac{u^2du}{u^2}\longrightarrow\infty!$$ (like $\int^{\Lambda_c}du\propto\Lambda_c$, so called a "linear divergence")

How do we deal with this?

Note that it came from having $V(\vec{x}_1-\vec{x}_2)=\lambda\,\delta^3(\vec{x}_1-\vec{x}_2)$.

Is it enough to take $\delta^3\to e^{-|\vec{x}_1-\vec{x}_2|^2/b^2}$?

- would be finite, but sensitive to b...

No! But we have to renormalize for free-space scattering, and this fixes the finite density result as well!