$$iG_0(\tilde{k})_{\alpha\beta} = i\delta_{\alpha\beta}\left(\frac{\Theta(|\vec{k}|-k_f)}{k_0-w_k+i\varepsilon} + \frac{\Theta(k_f-|\vec{k}|)}{k_0-w_k-i\varepsilon}\right)$$

$$w_k \equiv \frac{k^2}{2m}$$

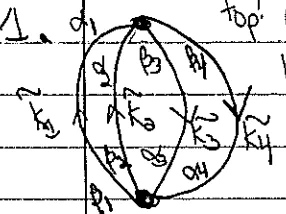
①

10/22/14

Wednesday 8805 Class

- On Monday we looked at the Feynman rules for the energy density in momentum space at $T=0$

$\tilde{k} \equiv (k_0, \vec{k})$

1. 

top: $(\tilde{k}_1 + \tilde{k}_2)_{in} = (\tilde{k}_3+\tilde{k}_4)_{out} \Rightarrow \tilde{k}_1 \to \tilde{p},\ \tilde{k}_2 \to \tilde{k},\ \tilde{k}_3 \to \tilde{k}+\tilde{q},\ \tilde{k}_4 \to \tilde{p}-\tilde{q}$

bottom: $(\tilde{k}_1+\tilde{k}_2)_{out} = (\tilde{k}_3+\tilde{k}_4)_{in}$

$\Rightarrow \tilde{p}, \tilde{k}, \tilde{q}$ are the independent momenta

2. $$iG^0_{\alpha_1\beta_1}(\tilde{p})\, iG^0_{\alpha_2\beta_2}(\tilde{k})\, iG^0_{\alpha_3\beta_3}(\tilde{k}+\tilde{q})\, iG^0_{\alpha_4\beta_4}(\tilde{p}-\tilde{q}) \times (-iC_0)^2$$

$$\times \underbrace{\left(\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4} + \delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3}\right)}_{\leftarrow\text{top}\rightarrow}\underbrace{\left(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2} + \delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1}\right)}_{\leftarrow\text{bottom}\rightarrow}$$

3. $$\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}\left(\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4} + \delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3}\right)\left(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2} + \delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1}\right)$$

$= 2\nu(\nu-1)$ ← check $\bigcirc\!\!-\!\!\bigcirc \propto (-\nu)^2$ $\ominus \propto -\nu$

4. $$\int\frac{d^4\tilde{k}}{(2\pi)^4}\int\frac{d^4\tilde{p}}{(2\pi)^4}\int\frac{d^4\tilde{q}}{(2\pi)^4}$$ Is this the best choice of momenta?

5. $S=2$ (permute → can't tell switch), 2 2-tuples $\Rightarrow (2!)^2 \Rightarrow 1/8$

$$\Rightarrow \mathcal{E}_2 = -i\frac{C_0^2}{4}\nu(\nu-1)\int\frac{d^4\tilde{k}}{(2\pi)^4}\int\frac{d^4\tilde{p}}{(2\pi)^4}\int\frac{d^4\tilde{q}}{(2\pi)^4}\, G^0(\tilde{p})\,G^0(\tilde{k})\,G^0(\tilde{k}+\tilde{q})\,G^0(\tilde{p}-\tilde{q})$$

$$\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\overset{ⓐ}{\left[\frac{\Theta(|\vec{p}|-k_f)}{p_0-w_p+i\varepsilon}+\frac{\Theta(k_f-|\vec{p}|)}{p_0-w_p-i\varepsilon}\right]}\overset{ⓑ}{\left[\frac{\Theta(|\vec{k}|-k_f)}{k_0-w_k+i\varepsilon}+\frac{\Theta(k_f-|\vec{k}|)}{k_0-w_k-i\varepsilon}\right]}\overset{ⓒ}{\left[\frac{\Theta(|\vec{k}+\vec{q}|-k_f)}{k_0+q_0-w_{\vec{k}+\vec{q}}+i\varepsilon}+\frac{\Theta(k_f-|\vec{k}+\vec{q}|)}{k_0+q_0-w_{\vec{k}+\vec{q}}-i\varepsilon}\right]}$$

$$\times\overset{ⓓ}{\left[\frac{\Theta(|\vec{p}-\vec{q}|-k_f)}{p_0-q_0-w_{\vec{p}-\vec{q}}+i\varepsilon}+\frac{\Theta(k_f-|\vec{p}-\vec{q}|)}{p_0-q_0-w_{\vec{p}-\vec{q}}-i\varepsilon}\right]}$$

10/22/14

(2)

$$(a)\times(d)\quad \int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\left[\frac{\theta(|\vec p|-k_F)\,\theta(k_F-|\vec p-\vec q|)}{(p_0-w_p+i\epsilon)(p_0-q_0-w_{|\vec p-\vec q|}-i\epsilon)}+\frac{\theta(k_F-|\vec p|)\,\theta(|\vec p-\vec q|-k_F)}{(p_0-w_p-i\epsilon)(p_0-q_0-w_{\vec p-\vec q}+i\epsilon)}\right]$$ ← Why only these two?

$$=\frac{1}{\cancel{2\pi}}\left[\frac{-\cancel{2\pi}i\ (e)\ \theta(|\vec p|-k_F)\,\theta(k_F-|\vec p-\vec q|)}{w_p-q_0-w_{|\vec p-\vec q|}-i\epsilon}+\frac{\cancel{2\pi}i\ (f)\ \theta(k_F-|\vec p|)\,\theta(|\vec p-\vec q|-k_F)}{w_p-q_0-w_{\vec p-\vec q}+i\epsilon}\right]$$ ← $q_0=w_p-w_{p-q}\pm i\epsilon$

$$(b)\times(c)=\frac{1}{\cancel{2\pi}}\left[\frac{-\cancel{2\pi}i\ (g)\ \theta(|\vec k|-k_F)\,\theta(k_F-|\vec k+\vec q|)}{w_k+q_0-w_{|\vec k+\vec q|}-i\epsilon}+\frac{\cancel{2\pi}i\ (h)\ \theta(k_F-|\vec k|)\,\theta(|\vec k+\vec q|-k_F)}{w_k+q_0-w_{\vec k+\vec q}+i\epsilon}\right]$$

$p\to k$
$p-q\to k+q$

$q_0=w_{\vec k+\vec q}-w_{\vec k}\mp i\epsilon$

clockwise

$$(f)\times(h)+(e)\times(g)=\frac{(2\pi i)}{2\pi}\cdot(i)^2\,\frac{\overbrace{\theta(k_F-|\vec p|)\,\theta(k_F-|\vec k|)}^{\text{holes}}\ \overbrace{\theta(|\vec p-\vec q|-k_F)\,\theta(|\vec k+\vec q|-k_F)}^{\text{particles}}}{w_p+w_k-w_{\vec p-\vec q}-w_{\vec k+\vec q}+i\epsilon}$$ ← consistent from both pieces

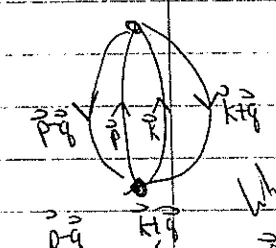
$$+\frac{(2\pi i)}{\cancel{2\pi}}(-i)^2\,\frac{\theta(|\vec p|-k_F)\,\theta(|\vec k|-k_F)\,\theta(k_F-|\vec p-\vec q|)\,\theta(k_F-|\vec k+\vec q|)}{w_p+w_k-w_{\vec p-\vec q}-w_{\vec k+\vec q}-i\epsilon}$$

If we could switch $p\leftrightarrow p-q$, $k\leftrightarrow k+q$ these would be the same! ✓

$\vec p\,'=\vec p-\vec q$, $\vec k\,'=\vec k+\vec q$, $\vec q\,'=-\vec q$ $\Rightarrow$ $\vec p=\vec p\,'+\vec q=\vec p\,'-\vec q\,'$ and $\vec k=\vec k\,'+\vec q\,'$

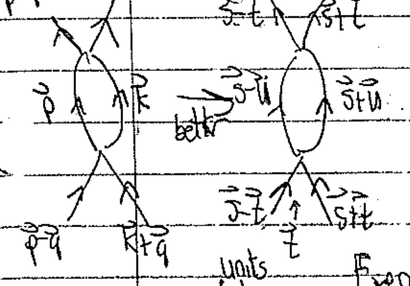
$\Rightarrow$ a factor of 2

$$\Rightarrow\ \mathcal{E}_2=\underbrace{(-i)\frac{c_0^2}{4}\nu(\nu-1)(-2i)}_{-2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\,\frac{\theta(|\vec p|-k_F)\,\theta(|\vec k|-k_F)\,\theta(k_F-|\vec p-\vec q|)\,\theta(k_F-|\vec k+\vec q|)}{w_p+w_k-w_{\vec p-\vec q}-w_{\vec k+\vec q}-i\epsilon}$$



rather asymmetric! But note: sum of particle energies − sum of hole energies } we'll see this again

What is dependence on k_F? Scale all variables by k_F → $k_F^9/k_F^2=k_F^7$



$k_F(\vec s-\vec u)=\vec p$
$k_F(\vec s+\vec u)=\vec k$
$k_F(\vec s-\vec t)=\vec p-\vec q$
$k_F(\vec s+\vec t)=\vec k+\vec q$

$\frac{1}{4}\cdot 2\cdot 8$

$$\Rightarrow\ \boxed{\mathcal{E}_2=-4c_0^2M\nu(\nu-1)k_F^7\int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\int\frac{d^3u}{(2\pi)^3}\frac{1}{u^2-t^2-i\eta}\times\theta(|\vec s-\vec u|-1)\,\theta(|\vec s+\vec u|-1)\,\theta(1-|\vec s+\vec t|)\,\theta(1-|\vec s-\vec t|)}$$

uses $\frac{1}{2m}\left[(\vec s-\vec u)^2+(\vec s+\vec u)^2-(\vec s-\vec t)^2-(\vec s+\vec t)^2\right]=\frac{2}{2m}(u^2-t^2)$

Exercise: Jacobian is $8k_F$

sharp: $C_0(\Lambda_c) = \frac{4\pi a_0}{M}\left(1+\frac{2}{\pi}a_0\Lambda_c\right) = C_0^{(1)} + C_0^{(2)} \Rightarrow C_0^{(2)} = [C_0^{(1)}]^2 M \frac{\Lambda_c}{2\pi^2}$

 $= \frac{C_0}{2}\left(1-\frac{1}{\nu}\right)\rho^2$

10/22/14

Aside: Do the Jacobian for switching from $\vec{p}, \vec{k}, \vec{q}$ to $\vec{s}, \vec{t}, \vec{u}$

- Do the x-component and cube it.

$$\int dk_x\, dp_x\, dq_x = \int ds_x\, dt_x\, du_x \begin{vmatrix} \frac{\partial k_x}{\partial s_x} & \frac{\partial k_x}{\partial t_x} & \frac{\partial k_x}{\partial u_x} \\ \frac{\partial p_x}{\partial s_x} & \frac{\partial p_x}{\partial t_x} & \frac{\partial p_x}{\partial u_x} \\ \frac{\partial q_x}{\partial s_x} & \frac{\partial q_x}{\partial t_x} & \frac{\partial q_x}{\partial u_x} \end{vmatrix} \to k_f^3 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 2k_f^3 \Rightarrow 8k_f^9 \text{ total}$$

$\vec{p} = k_f(\vec{s}-\vec{u})$ $\quad \vec{q} = k_f(\vec{t}-\vec{u})$

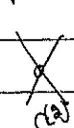
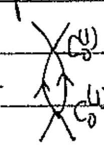
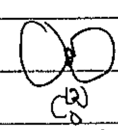
$\vec{k} = k_f(\vec{s}+\vec{u})$

- The role that we get particle energies − hole energies will be immediate from the Goldstone diagram rules!
  - Note that the pole is never hit, although right at the Fermi surface it is close, ⇒ real.
- The integral that is left is divergent
  - The Θ functions restrict $|\vec{s}|$ and $|\vec{t}|$ to be < 1, so these integrals are bounded
  - But the $\vec{u}$ integration runs to $|\vec{u}| \to \infty$. For large u, $\Theta(|\vec{s}\pm\vec{u}|-1)\,\Theta(|\vec{s}-\vec{u}|-1) = 1$ and $\frac{1}{u^2-t^2} \to \frac{1}{u^2}$, so the integral goes like $\int \frac{u^2 du}{u^2} \sim \int du \to \infty$ if unregulated.

- To isolate the divergence, write $\Theta(|\vec{s}\pm\vec{u}|-1) = 1 - \Theta(1-|\vec{s}\pm\vec{u}|)$
  - Then the terms with the $\Theta(1-|\vec{s}\pm\vec{u}|)$ are finite, so we only have to worry about the terms with 1's.

$$\Rightarrow \mathcal{E}_2^{\text{divergent}} \underset{\substack{\text{real part}\\ \text{of}}}{=} -4C_0^2 M \nu(\nu-1) k_f^7 \int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}(1+|\vec{s}+\vec{t}|)(1-|\vec{s}-\vec{t}|)\left[\int^{\Lambda_c}\frac{d^3u}{(2\pi)^3}\frac{1}{u^2-t^2-i\eta}\right]$$

The positive power of Λ_c is precisely cancelled when we use $C_0^{(2)}$ in 

That is,  +  = finite ⇒  +  is finite.

- In dimensional regularization with minimal subtraction, this is very clean: the real part of the divergent integral is zero.
  - Pauli blocking is IR (low k) while sensitivity to UV is near $\Lambda_c > k_f$

⇒ counterterms for UV in free space automatically renormalize the finite density parts. [But the distribution of strength depends on the regulator.]

10/22/14

Dilute expansion to k_f^6

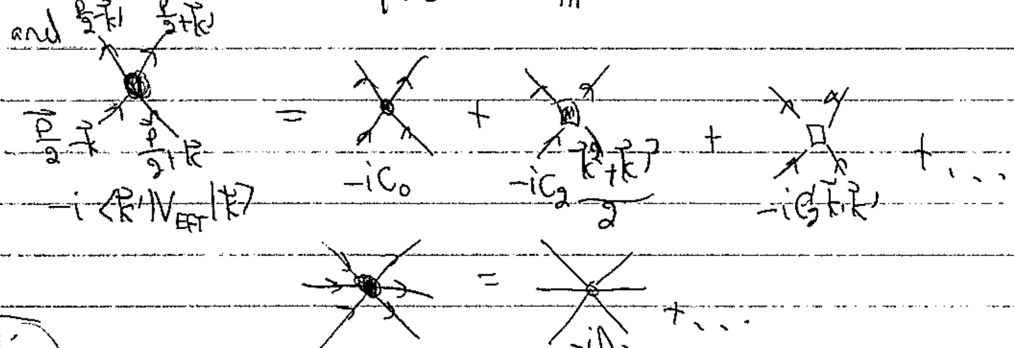
energy / particle:

$$\frac{E}{A} = \frac{k_f^2}{2M}\Big[\frac{3}{5} + (\nu-1)\Big\{\overset{(a)}{\frac{2}{3\pi}(k_f a_0)} + \overset{(b)}{\frac{4}{35\pi^2}(11-2\ln 2)(k_f a_0)^2} + \overset{(c)}{\frac{1}{10\pi}(k_f r_0)(k_f a_0)^2}$$
$$+ \big[\overset{(d)}{0.076} + \overset{(e)}{0.057}(\nu-3)\big](k_f a_0)^3\Big\} + \overset{(f)}{(\nu+1)\frac{1}{5\pi}(k_f a_p)^3}$$
$$+ (\nu-1)(\nu-2)\overset{(g)}{\frac{16}{27\pi^3}}(4\pi-3\sqrt{3})(k_f a_0)^4\ln(k_f a_0) + \dots\Big]$$

- Written this way it is an expansion in $k_f a_0$ or $k_f r_0$ (or maybe $\frac{k_f a_0}{\pi}$!).
  - but not a power series about $k_f = 0 \Rightarrow \ln(k_f a_s)$ term
- Derived with one diagram per term if we use DR/MS with

$$L_n \equiv \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{q^{2n}}{k^2-q^2+i\epsilon} \xrightarrow{D\to 4} -\frac{i}{4\pi}k^{2n+1}$$

and

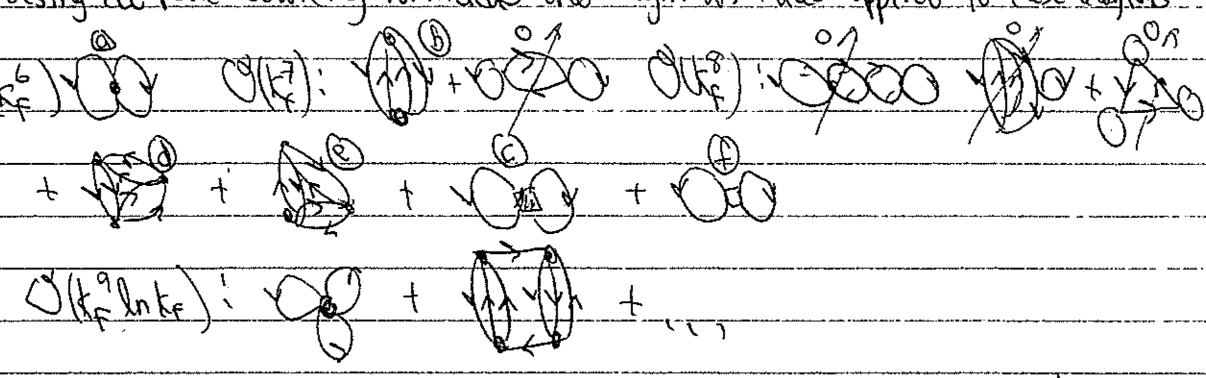


matching in free space scattering:

$$C_0 = \frac{4\pi a_0}{M}, \quad C_2 = C_0\frac{a_0 r_0}{2}, \quad C_2' = \frac{4\pi a_p^3}{M} \qquad k\cot\delta_0 = -\frac{1}{a_0} + \frac{1}{2}r_0 k^2 + \dots$$
$$k\cot\delta_1 = -\frac{3}{k^2 a_p^3} + \dots$$

energy density: Using the power counting formulas and Feynman rules applied to these diagrams:

$\mathcal{E} - \mathcal{E}_0 =$



k_f^3 higher than $\frac{E}{A}$

- In practice, we would typically have a cutoff regulator and non-contact terms $\rightarrow$ numerical evaluation. Could be perturbation theory or nonperturbative.

10/22/14

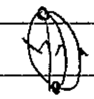
Power counting dilute system: very clean with DR/MS
[from Section 8 notes]

- If breakdown scale is Λ_b (e.g. $\Lambda_b \sim \frac{1}{a_0}, \frac{1}{r_0}$) then diagram with V_{2i}^n n-body vertices scales as $(k_F/\Lambda_b)^\beta$ where

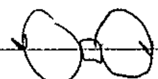
$$\boxed{\text{eq. (91) in sect. 4}} \quad \beta = 5 - \frac{3}{2}\underset{\substack{\uparrow \\ \text{external lines}}}{E} + \sum_{n=2}^{\infty} \sum_{i=0}^{\infty} (3n + 2i - 5) V_{2i}^n \quad \begin{array}{l} n \leftarrow \text{n-body vertex} \\ 2i \leftarrow 2i \text{ derivatives} \end{array}$$

- Now $n \geq 2$, $2i \geq 0$, $V_{2i}^n \geq 1 \Rightarrow (3n+2i-5)V_{2i}^n \geq 1 \Rightarrow \beta \geq 6$ guaranteed!

 $\Rightarrow V_0^2 = 1 \Rightarrow \beta = 5 + (3\cdot 2 + 2\cdot 0 - 5)\cdot 1 = 6 \quad \mathcal{O}(k_F^6)$

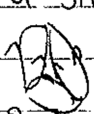
 $\Rightarrow V_0^2 = 2 \Rightarrow \beta = 5 + (3\cdot 2 + 2\cdot 0 - 5)\cdot 2 = 7 \quad \mathcal{O}(k_F^7)$

- switch vertex for one with more derivatives

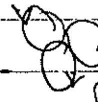
  $\Rightarrow 2i\uparrow$ implies $\beta\uparrow$ always

$C_0 \Rightarrow \beta = 6 \qquad C_2(k^2 + k'^2) \Rightarrow 2i = 2 \Rightarrow \beta = 8$

- add a similar vertex

$V_0^2 = 3 \Rightarrow \beta = 8 \qquad V_0^2 = 4 \Rightarrow \beta = 9 \quad \Rightarrow V_{2i}^n \uparrow$ implies $\beta\uparrow$ always

- 3-body  $\propto k_F^6 \propto \rho^2$ vs.  $\propto \rho^3 \propto k_F^9$
[$n=3, i=0, V_0^3 = 1$]

$\Rightarrow$ only a finite # of diagrams at each order guaranteed $\Rightarrow$ EFT systematic
- again, not a power series

- Is this like low-density neutron matter?
  - no: if $a_0 \gg r_0$ then must sum all diagrams with C_0 and include pairing $\Rightarrow$ numerical nonperturbative calculation

10/22/14

- We've looked in some detail at one example of Feynman rules. There are many alternatives (coordinate space, Euclidean, Goldstone, different basis) and generalizations (finite range potentials, exchange of dynamical pions, electroweak).

- Let's consider some general features of these rules and see how they arise in a very simple model motivated by a general path integral formalism.

Common features of Feynman rules for diagrams:

* • Draw all possible topologically distinct diagrams (lines may be directed, i.e. one end is different from the other)
* • Connected diagrams only
* • lines from quadratic parts $\Rightarrow$ Green's functions $\equiv$ propagators
* • vertices have rules that come from the Lagrangian
* • symmetry factors adjust the combinatoric factors
** • sum over all intermediate thing that can happen, depending on the representation
  - space and time or momentum and frequency
  - spin and isospin
* • we can devise diagrammatic equations (Dyson equations) that sum up infinite subsets of the diagrams
* • we can have diagrams for quantities like the energy (or free energy) that will be closed, or for expectation values of operators, which will have free ends.

- With our model partition function we can see all of the starred features and a slight generalization to a matrix version even gets the double-starred feature.

- There are ~30 pages of old notes available. We will do some in class and other parts on Piazza.

10/22/14

Our model partition is just an integral

$$Z = \int d\xi \, e^{-f(\xi)} \longrightarrow Z_\lambda = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi/a}} e^{-\frac{a\xi^2}{2} - \frac{\lambda}{4}\xi^4}$$

[We use the notation here of Negele and Orland, which we'll see if not really "natural", but it keeps things consistent and probably highlights the consequences of choices better.]

- What does this have to do with what we've been talking about or plan to discuss? Just some teasers here...

• Recall that in statistical mechanics and thermodynamics the partition function is the key to all calculations.

• In the grand canonical ensemble,

$$Z_G \equiv \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu\hat{N})}$$

where $\beta = \frac{1}{k_B T}$, μ is the chemical potential, $\hat{H}$ is the Hamiltonian, and $\hat{N}$ is the total number op.

• The trace Tr is a sum of diagonal matrix elements in any complete basis.

• Let $\hat{H}' \equiv \hat{H} - \mu\hat{N}$ [or $\hat{H}' + \alpha\hat{O}$ in general, with $\hat{O}$ a Hermitian operator]

• If we use a complete set of eigenstates of $\hat{H}'$:

$$H'|\Psi_n\rangle = E_n'|\Psi_n\rangle \Rightarrow Z_G = \mathrm{Tr}\, e^{-\beta\hat{H}'} = \sum_n \langle\Psi_n|e^{-\beta\hat{H}'}|\Psi_n\rangle = \sum_n e^{-\beta E_n'} \underbrace{\langle\Psi_n|\Psi_n\rangle}_{1 \text{ if normalized}}$$

Then we recover the familiar form (For canonical ensemble, take $\hat{H}' \to \hat{H}$, $E_n' \to E_n$).

• So $\beta \to \infty \Rightarrow Z_G \underset{\beta\to\infty}{\to} e^{-\beta E_0'}$ and $-\frac{1}{\beta}\ln Z_G$ gives us the ground state energy in this limit (or free energy at $T\to 0$ $E - \mu N$)

• Expectation values follow from averaging over the partition function

take $\alpha = 0$ at end

$$\langle\hat{O}\rangle = \frac{\mathrm{Tr}(\hat{O}\, e^{-\beta\hat{H}'})}{\mathrm{Tr}\, e^{-\beta\hat{H}'}} = \frac{1}{Z_G}\mathrm{Tr}(\hat{O}\, e^{-\beta\hat{H}'}) = \frac{-\frac{1}{\beta}\frac{\partial}{\partial\alpha}\mathrm{Tr}(e^{-\beta(\hat{H}'+\alpha\hat{O})})}{\mathrm{Tr}\, e^{-\beta(\hat{H}'+\alpha\hat{O})}} = -\frac{1}{\beta}\frac{1}{Z_{G\alpha}}\frac{\partial Z_{G\alpha}}{\partial\alpha}$$

any operator

$$Z_{G\alpha} = \mathrm{Tr}\, e^{-\beta(\hat{H}'+\alpha\hat{O})} \qquad = -\frac{1}{\beta}\frac{\partial}{\partial\alpha}\ln Z_{G\alpha}$$

10/22/14

If we take $\hat{O} = \hat{N}$, then

$$\langle \hat{N} \rangle = \frac{\mathrm{Tr}\left(\hat{N}\, e^{-\beta(\hat{H}-\mu\hat{N})}\right)}{\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}} = \frac{\frac{1}{\beta}\frac{\partial}{\partial\mu}\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}}{Z_G} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_G$$

$$\left[\text{check: } \ln Z_G = -\beta\Omega \Rightarrow \langle \hat{N} \rangle = -\frac{1}{\beta}\frac{\partial}{\partial\mu}(\beta\Omega) = -\frac{\partial\Omega}{\partial\mu} = nV\right]$$

So how do we evaluate this partition function?

- We don't know the $|\Psi_n\rangle$'s, or we would already have solved the problem
- Direct numerical evaluation of the exponent in matrix form (discretized) is not feasible for any real case. But for our model, imagine one lattice point and $1 = \int_{-\infty}^{\infty} ds\, |s\rangle\langle s|$ as a complete set of states.
- More generally, we can only evaluate $e^{-\beta\hat{H}'}$ if β is very small, so we divide:

$$\mathrm{Tr}\, e^{-\beta\hat{H}'} = \mathrm{Tr}\big(\underbrace{e^{-\epsilon H'} e^{-\epsilon H'} \cdots e^{-\epsilon N'}}_{M \text{ copies}}\big) \qquad \beta = M\varepsilon$$

and insert complete sets of states between everyone.

- After a bunch of intermediate details, we end up with a path integral.
- The schematic form for fermions with a contact C_0 interaction is:

$$Z_G = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \int \overset{\swarrow \text{hides a lot!}}{\mathcal{D}(\psi,\psi^\dagger)}\; e^{-\int_{-\beta/2}^{\beta/2} d\tau \int d^3x\, \psi^\dagger_\alpha\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\psi_\alpha - \frac{C_0}{2}\psi^\dagger_\alpha\psi^\dagger_\beta\psi_\beta\psi_\alpha}$$

where the $\psi_\alpha(x)$'s are Grassman numbers.

If you squint, the model partition function looks (sort of) like a stripped down version of this expression. For now, you need to suspend your disbelief willingly ...

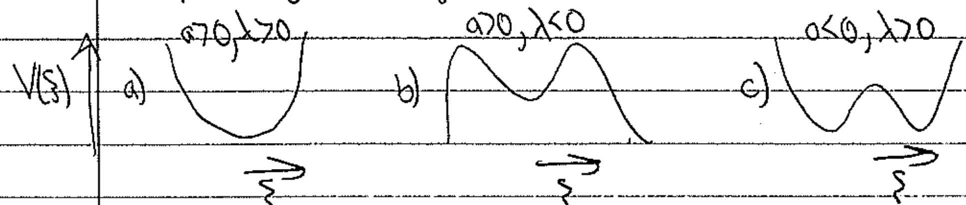
10/22/14

First pass at $Z_\lambda = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi/a}}\, e^{-\frac{a\xi^2}{2} - \frac{\lambda}{4}\xi^4}$

- More generally we will have $\xi \to \xi_1, \xi_2, \ldots$ and ξ_i and ξ_j coupled

$$\Rightarrow -\frac{1}{2}a\xi^2 \to -\frac{1}{2}\xi_i A_{ij}\xi_j \text{ with } \underline{\underline{A}} \text{ a matrix.}$$

- Depending on signs of a and λ, $V(\xi) = \frac{a\xi^2}{2} + \frac{\lambda}{4}\xi^4$ looks like



- We can consider various approximation strategies → we'll start with perturbation theory in λ
  - based on being able to do Gaussian integrals

$$\int_{-\infty}^{\infty} d\xi\; e^{-a\xi^2/2} = \sqrt{\frac{2\pi}{a}} \quad \text{(and generalized to } \xi_i A_{ij}\xi_j)$$

- Add a "source term" to the exponent

$$Z_\lambda \to Z_\lambda[j] = N\int_{-\infty}^{\infty} d\xi\; e^{-\frac{a\xi^2}{2} - \frac{\lambda\xi^4}{4} + j\xi} \qquad \text{(N to avoid distractions and see cancellation)}$$

so that $Z_\lambda = Z_\lambda[j]\big|_{j=0}$ ⇒ set $j=0$ at the end to recover the thing we want.

- cf. adding a magnetic field under your control.

Now $\frac{\partial}{\partial j} e^{j\xi} = \xi e^{j\xi} \Rightarrow f(\xi)e^{j\xi} = f\left(\frac{\partial}{\partial j}\right)e^{j\xi}$

where $f\left(\frac{\partial}{\partial j}\right)$ means to Taylor expand $f(\xi)$ and replace ξ by $\frac{\partial}{\partial j}$, acting to the right.

10/22/14

to complete the square

$$\Rightarrow Z_\lambda[j] = N e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} \int_{-\infty}^{\infty} d\xi\, e^{-\frac{a}{2}\xi^2 + j\xi} \quad \left(-\frac{a}{2}\frac{j^2}{a^2} + \frac{a}{2}\frac{j^2}{a^2}\right)$$

complete the square

$$= \left\{ e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j} \right\} \underbrace{N \int_{-\infty}^{\infty} d\xi'\, e^{-\frac{a}{2}\xi'^2}}_{Z_{\lambda=0} \equiv Z_0 \text{ independent of } j}$$

$$= Z_0\, e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}$$

non-interacting partition fn Z_0

and $\frac{Z_\lambda}{Z_0} = \left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j} \right]\Big|_{j=0}$

And $\ln Z/Z_0 \to \Omega - \Omega_0$ (like $E - E_0$ we've been calculating)

To evaluate to a desired order in λ, expand the first potential to that order.
- Then expand the second one just enough so that we get non-zero terms when $j = 0$.

$$\frac{Z_\lambda}{Z_0} = \left[1 - \frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4 + \frac{1}{2!}\left(\frac{\lambda}{4}\right)^2\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4 - \frac{1}{3!}\left(\frac{\lambda}{4}\right)^3\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4 + \ldots\right]$$
$$\times \left[1 + \frac{1}{2}(j a^{-1} j) + \frac{1}{2!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right) + \ldots\right]$$

- Do λ^1 and λ^2, for which Mathematica tells us $\frac{Z_\lambda}{Z_0} = 1 - \frac{3}{4}\frac{\lambda}{a^2} + \frac{105}{32}\frac{\lambda^2}{a^4} + \ldots$

nontrivial coefficients!

$$\lambda^1 : -\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4 \frac{1}{2!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right) = -\frac{\lambda}{a^2}\frac{1}{32}\left(\frac{\partial}{\partial j}\right)^4 (jjjj)$$

only surviving term as $j \to 0$

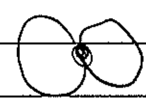
$$= -\frac{\lambda}{a^2}\frac{1}{32}4! = -\frac{3\lambda}{a^2} \checkmark$$

4! from first $\frac{\partial}{\partial j}$ can pick from 4 j's, next from 3 j's, next from 2 j's, last has no choice. So $4 \cdot 3 \cdot 2 \cdot 1 = 4!$

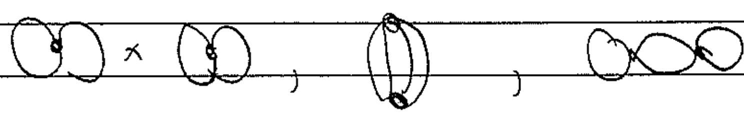
10/22/14

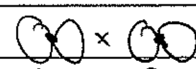
Basic calculation is $\frac{\partial}{\partial j} j = 1$. [for path integrals almost as simple!]

- associate $\bar{a}^{-1}$ in $\frac{1}{2} j \bar{a}^{-1} j$ with line •—• (inverse of quadratic operator ⇒ propagator) and the "interaction" $\frac{\lambda}{4!}$ with a •, then the result can be represented as



- At the next order, the diagrams are



- By considering $\ln \frac{Z_\lambda}{Z_0}$, we'll find  goes away and we'll see how to assign factors.