Wednesday 8805 Class

1. On Monday we looked at the Feynman rules for $K=f(x_0,\vec{x})$ energy density in momentum space at $T=0$

$$iG^0(q) = \frac{i\delta_{\mu\nu}\left((\vec{K}^2 - \vec{k}^2) + \vec{q} \cdot (\vec{K} - \vec{k})\right)}{k_0w_k+i\varepsilon}$$

2. $k_0 = \frac{E_k}{\sqrt{\sum_n^2}}$

3. $\vec{k}, \vec{q}$ are the independent momenta

$$iG^0(q) iG^0(t) iG^0(-t) iG^0(t-\vec{q}) \times (iC_0)^2$$

$$= (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho})(\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho})$$

$$\Rightarrow \text{check } 0 - 0 \times (y-y^2) \Rightarrow x - y$$

4. Is this the best choice of momenta?

5. $S=2$ (parity $\rightarrow$ ant $\rightarrow$ switch), 2 $\sigma$-types $\Rightarrow (21)^9 \Rightarrow \frac{1}{8}$

$$\Rightarrow \frac{1}{8} = -i \sum_{\sigma_1} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} G(\vec{q}) G(\vec{k}) G(\vec{K}) G(\vec{k}-\vec{q})$$

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \left[ \theta(\vec{q}^2 - k^2) + \theta(\vec{k}^2 - q^2) \right] \left[ \theta(k_0 - p^+ + \vec{k} \cdot \vec{r}) + \theta(q_0 - p^+ + \vec{q} \cdot \vec{r}) \right]$$

$$\times \left[ \frac{\theta(\vec{q}^2 - k^2)}{k_0w_k - i\varepsilon} + \frac{\theta(\vec{k}^2 - q^2)}{q_0w_q - i\varepsilon} \right]$$

$$\Rightarrow \text{evaluate integral}$$
\[
\text{(3) }
\]

Aside: Do the Jacobian for switching from \( \hat{r}, \hat{\theta}, \hat{\phi} \) to \( \hat{r}, \hat{\theta}, \hat{\phi} \)

- Do the x-component and cube it,

\[
\begin{array}{c|c|c|c}
\hat{r} &= k_0 \theta (\hat{\phi} - \hat{\phi}) & \theta &= 1 & \hat{\phi} \\
\hat{\theta} &= k_0 (\theta - \hat{\theta}) & \hat{\phi} &= \frac{1}{\sin \theta} & \hat{\phi} \\
\hat{\phi} &= k_0 \hat{\phi} & \hat{\phi} &= \theta & \hat{\phi} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hat{r} &= k_0 \hat{r} & \hat{\theta} &= k_0 \hat{\theta} & \hat{\phi} &= k_0 \hat{\phi} \\
\end{array}
\]

- The pole that we get particle energies -- hole energies will
  be immediate from the Goldstone diagram above.
- Note that the pole is never hit, although right at \( \Phi \) surface it is close, \( \Im \hat{\phi} \)
- The integral that is left is divergent
  - The \( \hat{\theta} \) functions restrict \( |\hat{r}| \) and \( |\hat{\phi}| \) to be < 1, so these
    integrals are bounded
  - But the \( \hat{\phi} \) integration runs to \( \hat{\phi} \rightarrow \infty \). For large \( \hat{\phi} \),
    \( \sin \hat{\phi} \rightarrow 1 \) and \( \hat{\phi} \rightarrow \hat{\phi} \), so \( \hat{\phi} \)
    integral goes like \( \sin \hat{\phi} - \hat{\phi} \rightarrow \infty \) if unregulated.

- To isolate the divergence, write \( \sin (\theta - \hat{\phi}) = 1 - \theta (1 - \hat{\phi}) \)
  - Then its terms with \( \theta \) \( \theta (1 - \hat{\phi}) \) are finite, so we
    only have to worry about the terms with \( \hat{\phi} \).

\[
E_{\text{divergent}} = -4C_0^2 \left( \begin{array}{c} 1 \\ m \end{array} \right)^2 \hat{r} \theta \sin \theta \left[ \frac{\hat{\phi}^3}{\sin^3 \theta} \left( 1 - \frac{\sin^2 \theta}{\sin^2 \phi} \right) \right]
\]

- The positive power of \( \hat{\phi} \) is precisely cancelled when we use \( C_0^2 \) in \( \hat{r} \theta \)

That is, \( \hat{r} \theta \sin \theta \frac{\hat{\phi}^3}{\sin^3 \theta} \left( 1 - \frac{\sin^2 \theta}{\sin^2 \phi} \right) \) is finite.

- In dimensional regularization with minimal subtraction, this is very clean.
  - The real part of the divergent integral is zero.
  - Pauli blocking is \( K \) (low \( k \)), while sensitivity to \( \Phi \) is near \( \Lambda < k \).

\[
- \text{count the terms for } \Phi \text{ in free space automatically renormalize } \Lambda
\]

finite density puts. But the distribution of strength depends on

\( \chi \) regulator.
Dilute expansion to $\mathcal{O}(\mathcal{K}_\mathcal{F}^6)$

\[
\frac{E}{\mathcal{F}_\mathcal{C}} = \frac{\hbar^2}{2m} \left[ \frac{3}{2} (2\pi-0)(k_{\mathcal{F}_\mathcal{C}})^2 + \frac{1}{3}(3\pi^2/2\pi)(k_{\mathcal{F}_\mathcal{C}})^3 + \left( \frac{3\pi}{2} \right)^2 (k_{\mathcal{F}_\mathcal{C}})^4 \right]
\]

\[
+ \left( \frac{3\pi}{2} \right)^3 (k_{\mathcal{F}_\mathcal{C}})^5 \left( \ln(k_{\mathcal{F}_\mathcal{C}}) \right)^2 + \ldots
\]

Written this way, it is an expansion in $k_{\mathcal{F}_\mathcal{C}}^2$ or $k_{\mathcal{F}_\mathcal{C}}^4$ (or maybe $k_{\mathcal{F}_\mathcal{C}}^6$).

But not a proper series; it doesn't have $n$-point singularities.

Derived with one diagram per term. To use DR/MS, write

\[
\mathcal{L}_{\mathcal{N}} = \sum_{\alpha=1}^{\mathcal{N}} \frac{\alpha_{\mathcal{F}_\mathcal{C}}^2}{(2\mathcal{N}^2+1)} \frac{\mathcal{D}_{\mathcal{F}_\mathcal{C}}^2}{(2\mathcal{N}^2+1)} \mathcal{F}_\alpha^2
\]

and $\mathcal{F}_\mathcal{C}^2 = \mathcal{F}_\mathcal{C}^2$

\[
\mathcal{C}_\mathcal{F} = \frac{4\pi^2 \mathcal{C}_0}{m}, \quad \mathcal{C}_0 = \mathcal{C}_0 \frac{\mathcal{F}_\mathcal{C}^2}{2}, \quad \frac{\mathcal{F}_\mathcal{C}^2}{m} \quad \text{and} \quad \mathcal{F}_\mathcal{C}^2 \rightarrow \mathcal{F}_\mathcal{C}^6.
\]

Energy bounds: Using the power counting formulas and Feynman rules applied to these diagrams:

\[
\mathcal{F}_\mathcal{C}^6 = 0, \quad \mathcal{F}_\mathcal{C}^4 = 0, \quad \frac{\mathcal{F}_\mathcal{C}^2}{m} = 0
\]

In practice, we would typically have a cutoff regulator and non-contact terms $\sim$ numerical evaluation. Could be perturbation theory or non-perturbative.
Power counting dilute system: very clean, with no messy terms.

If breakdown scale is $\Lambda$, (e.g., $\Lambda_{\nu} \approx \frac{1}{a_{0}^{2} f_{0}}$) renormalization

with $V_{n}$, $n$-body vertex scales as $(k f_{\nu})^{n}$ where

$$\beta = 5 - \frac{3}{2} \epsilon + \sum_{n = 2}^{\infty} \sum_{i = 0}^{\infty} (3n + 2i - 5) V_{n} \sim \epsilon^{i} \text{derivative}$$

Now $n > 2, \exists i > 0$, $V_{n} > 1 \Rightarrow (3n + 2i - 5) V_{n} \sim 1 \Rightarrow \beta > 6$

$$\Rightarrow V_{0} = 1 \Rightarrow \beta = 5 + (3 \cdot 2 + 2 \cdot 0 - 5) \cdot 1 = 6 \Rightarrow \mathcal{O}(\epsilon)$$

$$\Rightarrow V_{0} = 2 \Rightarrow \beta = 5 + (3 \cdot 2 + 2 \cdot 0 - 5) \cdot 2 = 7 \Rightarrow \mathcal{O}(\epsilon^{2})$$

Switch vertex for one with more derivatives

$$0 \Rightarrow 0 \Rightarrow 0 \Rightarrow 0 \Rightarrow x \Rightarrow \exists i \Rightarrow \beta = i$$

Add a similar vertex

$$V_{0} = 3 \Rightarrow \beta = 8 \Rightarrow V_{2} = 4 \Rightarrow \beta = 9 \Rightarrow V_{3} \sim \epsilon \Rightarrow \beta \sim \epsilon$$

3-body $0 \times k f_{\nu} \times f_{\nu}$ vs. $0 \times f_{\nu} \times f_{\nu}$

$[\mu = 3, \lambda = 0, \nu = 1]$

$\Rightarrow$ only a finite # of diagrams at each order guaranteed $\Rightarrow$ EFT systematic.

Again, not a power series.

Is this like low-density neutron matter?

No: if $g_{0} > g_{0}^{\text{crit}}$ then we must sum all diagrams with $g_{0}$ and include pairing $\Rightarrow$ numerical perturbative calculation.
We've looked in some detail at one example of Feynman rules. There are many alternatives (coordinate space, Euclidean, Goldstone, different basis) and generalizations (finite range potentials, exchange of dynamical pairs, etc.etc.).

Let's consider some general features of these rules and see how they arise in a very simple model, motivated by a general path integral formalism.

Common features of Feynman rules for diagrams:

* Draw all possible topologically distinct diagrams (lines may be directed, i.e. one end is different from the other)
* Connected diagrams only
* Lines from quadratic parts \( \Rightarrow \) Green's functions \( \Rightarrow \) propagators
* Vertices have rules that come from the Lagrangian
* Symmetry factors adjust the combinatoric factors
* Sum over all intermediate thing that can happen, depending on the representation
  - Space and time or momentum and frequency
  - Spin and isospin
* We can devise diagrammatic equations (Dyson equations) that sum up infinite subsets of the diagrams
* We can have diagrams for quantities like the energy (or free energy) that will be closely, or far expectation values of operators, which will have fine ends.

With our model partition function we can see all of the starred features, and a slight generalization to a matrix version even gets the double-starred feature.

There are 30 pages of old notes available. We will do some in class and other parts on Piazza.
Our model partition is just an integral
\[ Z = \int d\psi e^{-f(\psi)} \rightarrow Z_1 = \sum_{\phi_0} e^{-\frac{g_0^2}{2\phi_0^2} - \frac{\Delta}{\phi_0^4}} \]

We use the notation here of Negele and Orland, which we'll see if not really "natural," but it keeps things consistent and probably highlights the consequences of choices better.

What does this have to do with what we've been talking about or plan to discuss? Just some treasure here...

- Recall that in statistical mechanics and thermodynamics the partition function is the key to all calculations.
  - In the grand canonical ensemble,
    \[ Z_G = \text{Tr} e^{-\beta \hat{H} + \beta \hat{\mu} \hat{N}} \]
    where \( \beta = \frac{1}{kT} \), \( \mu \) is the chemical potential, \( \hat{H} \) is the Hamiltonian, and \( \hat{N} \) is the total number op.
  - The trace \( \text{Tr} \) is a sum of diagonal matrix elements in any complete basis.
  - Let \( \hat{H}' = \hat{H} - \mu \hat{N} \) [or \( \hat{H} + \alpha \hat{O} \) in general, with \( \hat{O} \) an operator]
  - If we use a complete set of eigenstates of \( \hat{H}' \):
    \[ \hat{H}'|n\rangle = E_n'|n\rangle \Rightarrow Z_G = \text{Tr} e^{\beta \hat{H}' - \beta \mu \hat{N}} = \sum e^{\beta E_n} \langle n|e^{\beta \hat{O}}|n\rangle = \sum e^{\beta E_n} \langle n|e^{\beta \hat{O}}|n\rangle \]
  - In the canonical ensemble, we recover the familiar form (for canonical ensemble, take \( \hat{H}' \rightarrow \hat{H} \), \( E_n \rightarrow E_n \)).
    - So \( \beta \rightarrow \infty \Rightarrow Z_G \rightarrow \sum \text{e^{-\frac{E_n}{kT}}} \) and \( -\frac{1}{\beta} \ln Z_G \) gives us the ground state energy in this limit (in free energy \( E - \mu N \)).

- Expectation values follow from averaging over the partition function:
  \[ \langle \hat{O} \rangle = \frac{\text{Tr}(\hat{O} e^{\beta \hat{H}'})}{\text{Tr} e^{\beta \hat{H}'}} = \frac{1}{Z_G} \text{Tr} (\hat{O} e^{\beta \hat{H}'}) = \frac{1}{Z_G} \frac{\text{d}Z_G}{\text{d}\beta} \frac{\text{d} \text{Tr}(e^{\beta \hat{H}' + \alpha \hat{O}})}{\text{d} \alpha} = \frac{1}{Z_G} \frac{\text{d}Z_G}{\text{d} \alpha} \text{Tr}(e^{\beta \hat{H}' + \alpha \hat{O}}) \]
  \[ \frac{\text{d}Z_G}{\text{d} \alpha} \text{Tr}(e^{\beta \hat{H}' + \alpha \hat{O}}) = -\frac{1}{Z_G} \frac{\text{d}Z_G}{\text{d} \alpha} \text{Tr}(e^{\beta \hat{H}'}) \]
If we take \( \hat{\theta} = \hat{N} \cdot \theta \tan \theta \)

\[
\langle \hat{N} \rangle = \frac{\text{Tr} \left( \hat{N} e^{\hat{\theta} (\hat{A} - \mu \hat{N})} \right)}{\text{Tr} \, e^{\hat{\theta} (\hat{A} - \mu \hat{N})}} = \frac{\frac{\delta}{\partial \mu} \text{Tr} \, e^{\hat{\theta} (\hat{A} - \mu \hat{N})}}{Z_0} = \frac{1}{\exp(\theta \mu) \ln Z_0}
\]

[check: \( \ln Z_0 = -p \Omega \Rightarrow \langle \hat{N} \rangle = -\frac{\delta}{\partial \mu} \exp(\theta \mu) = -\frac{\mu}{\theta} = 1 / \Omega \)]

So how do we evaluate this partition function?

- We don't know \( \mu \) in this case, or we would already have solved the problem.

- Direct numerical evaluation of the exponent in matrix form (discretized) is not feasible for any real case, but for our model, imagine a lattice point and \( \mu = \text{fixed} \) as a complete set of states.

- More generally, we can only evaluate \( e^{\hat{\theta} A} \) if \( \theta \) is very small.

So we divide:

\[
\text{Tr} \, e^{\hat{\theta} A} = \frac{\text{Tr} \left( e^{\hat{\theta} A} \cdots e^{\hat{\theta} N} \right)}{\text{Tr} \, e^{\hat{\theta} N}} \theta \to 0 \epsilon
\]

and insert complete set of states between everything.

- After a bunch of intermediate details, we end up with a path integral.

- The schematic form for fermions with a contact \( \mu \) interaction is:

\[
Z_0 = \text{Tr} \, e^{\hat{\theta} (\hat{N} - \mu \hat{N})} = \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \, \exp \left( -\frac{1}{2} \int \bar{\psi} \left( \frac{1}{2} \partial^2 + \frac{1}{2} \partial^2 \right) \psi + \bar{\psi} \hat{a} \psi \right)
\]

where the \( \psi \) 's are Grassman numbers.

(Staff)

If you squint, the model partition function looks like a stripped down version of this expression. For now, you need to suspend your disbelief willingly...
First pass at \( z_\lambda = \int_0^\infty \frac{ds}{s^{a_3}} e^{-\frac{a_3}{2}s^4} \).

- More generally we will have \( \xi \Rightarrow \xi_1, \xi_2, \ldots \) and \( f: \xi \mapsto \xi_1 \), coupled
  \[ \Rightarrow -\frac{1}{2} a_3 \xi^2 \Rightarrow -\frac{1}{2} \sum_{i,j} A_{ij} \xi_i \xi_j \text{ with } A \text{ a matrix.} \]
- Depending on signs of \( a_3 \) and \( \lambda \), \( V(f) = \frac{a_3}{2} \xi^2 + \frac{1}{4} \xi^4 \) looks like

\[ \begin{matrix}
0, a_3 > 0 & 0, a_3 < 0 \\
0, a_3 < 0 & a_3 > 0
\end{matrix} \]

- We can consider various approximation strategies - well
  start with perturbation theory in \( \lambda \)
  - based on being able to do Gaussian integrals
    \[ \int_0^\infty s^{a_3/2} e^{-\frac{a_3}{2}s^4} = \frac{\Gamma(\frac{a_3}{2})}{\frac{a_3}{2}^{\frac{a_3}{2}}} \] (and generalized to \( \sum_{i,j} A_{ij} \xi_i \xi_j \))
  - Add a "source term" to the exponent
    \[ z_\lambda \Rightarrow z_\lambda \xi^j = N \int_0^\infty ds \ e^{-\frac{a_3}{2}s^4 + \xi^j s^2} \]
    \( N \) to avoid distractions.

- Add a magnetic field under your control:

\[ \frac{\partial}{\partial t} \xi_i = \xi_i \Rightarrow f(\xi)e^{\frac{1}{2} \xi^2 \phi} \]

- Using \( f(\xi) \) means to Taylor expand \( f(\xi) \) and replace \( \xi \) by \( \xi e^{\frac{1}{2} \xi^2 \phi} \)
10/22/14

\[ Z_{ij} = N e^{-\frac{2}{\sigma^2} \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right)} e^{-\frac{(y_i - y_j)^2}{2 \sigma^2}} \]

\[ \text{To evaluate up to a desired order in } \lambda, \text{ expand the field potential twice, once for each field.} \]

\[ Z_{ij} = Z_0 \left( \frac{2}{\sigma^2} \right)^{\frac{y_i^2 + y_j^2}{2}} \int e^{-\frac{2}{\sigma^2} \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right)} d^3 \phi \]

\[ = Z_0 \left( \frac{2}{\sigma^2} \right)^{\frac{y_i^2 + y_j^2}{2}} \]

\[ \text{evaluate partition function} \]

\[ \frac{Z_j}{Z_0} = \left[ e^{4 \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right)} \right]_{j=0} \]

\[ = \left[ e^{4 \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right)} \right]_{j=0} \]

\[ \ln Z_{ij} \to \lambda \to 0 \] (like \( \lambda \to 0 \) we've been calculating)

To evaluate up to a desired order in \( \lambda \), expand the field potential twice, once for each field.

Then expand the second once just enough so that we get non-zero terms when \( j=0 \).

\[ \frac{Z_j}{Z_0} = \left[ 1 + \frac{1}{2} \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right) + \ldots \right] \times \left[ 1 + \frac{1}{2} \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right) + \ldots \right] \]

Do \( \lambda^2 \) and \( \lambda^4 \), for which Mathematica tells us \( \frac{Z_j}{Z_0} = 1 - \frac{3}{4} \lambda^2 + \frac{3}{8} \frac{\lambda^4}{\sigma^2} + \ldots \)

\[ \lambda^4 = -\frac{1}{4} \lambda^2 \left( \frac{y_i^2}{2} + \frac{y_j^2}{2} \right) \frac{y_j^2}{2} \frac{y_j^2}{2} \frac{y_j^2}{2} \frac{y_j^2}{2} \frac{y_j^2}{2} \frac{y_j^2}{2} \frac{y_j^2}{2} \frac{y_j^2}{2} \]

only surviving coefficient:

\[ \text{form as } j \to 0 \]

\[ \lambda^4 \text{ from first 3} \]

\[ \text{can pick from 4 j's, not from 3 j's, not from 2 j's, last has no choice. So } 4, 3, 2, 1 = 4. \]
Basic calculation is $\delta_{ij} = 1$. For path integrals almost as simple,

- Associate $\bar{a}^i \in \delta_j^j$ with line $\rightarrow$ (inverse of quadratic operator $\Rightarrow$ propagator) and the "interaction" $\cdot$ with $\cdot$.

Then the result can be represented as

- At the next order, the diagrams are

  $\bigcirc \times \bigcirc$, \( \cdot \), \( \bigcirc \bigcirc \bigcirc \)

- By considering $\ln \frac{3}{3}$, we'll find $\bigcirc \times \bigcirc$ goes away and we'll see how to assign factors.