Model partition function: general form is $Z = \int \exp \left( - \frac{S}{\beta} \right)$

- Analogies to EFT:
  - At low "energies" we have an expansion of $f$ in $\mu$ in form of a Taylor expansion:
  $$f(\mu) = f(0) + \frac{\mu}{2!} f''(0) + \ldots.$$
  - Now suppose we know from physics reasons there is a symmetry under $\mu \to -\mu$ (that is $Z$ is unchanged), which will be
    introduced by the low-energy limit $\mu \to 0$ only even terms in $f(\mu)$.
    - No constant term is additive in $\mu$ and doesn't change
    - Physics $\rightarrow$ general expansion to quartic order is

$$Z = \sum_{\mu = 0}^{\infty} \frac{\alpha^\mu}{\mu!} e^{-\frac{\mu^2}{4}} [Z, \text{ because approximation as function of } \lambda \text{ will be explored}].$$

- Background on partition functions from 10/22/14 (3-5) notes.
- Continue with 10/22/14 (9-11)

- Generalized Gaussian result (on paper)

$$\int \prod_{i} d \phi_i e^{-\frac{1}{2} \phi_i^T A \phi_i + \frac{1}{2} \sum_{i} F_i \phi_i} = \int d \psi d \phi e^{-\frac{1}{2} \psi^T A \psi + \frac{1}{2} \sum_{i} F_i \psi}$$

$$= (2\pi)^{n/2} (\det A)^{-1/2} e^{\frac{1}{2} \sum_{i} F_i}$$

- Let's continue to higher order.
\[ \psi(\gamma) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]

- Many combinatoric factors, but we can organize according to how the \( j^i \) attacks the \( j^i j \) terms, as represented by diagrams:

- For each \( j^i \), there are two \( j^i \) terms:
  - Each \( j^i \) hits two separate \( j^i j \) terms.
  - Each \( j^i \) picks up one \( j^i j \) term from each \( j^i j \) term.

- For each \( j^i \), the \( j^i \) terms hit one \( j^i j \) term and need to be set to zero.

- Now what if we want an expectation value, such as \( \langle f^{j_0} \rangle \):

\[ \langle f^{j_0} \rangle = \frac{\text{N}}{\Sigma_{j_0} \psi(\gamma)^{j_0} e^{-\frac{j_0^2}{2}}} = \left( \frac{\psi(\gamma)^{j_0} e^{-\frac{j_0^2}{2}}}{\text{N}} \right)_{j_0} \]

\[ \text{Ns cancel} = \left( \frac{\psi(\gamma)^{j_0} e^{-\frac{j_0^2}{2}}}{\text{N}} \right)_{j_0} \]

- We'll carry this out to a couple of orders, but first let's recall that for thermodynamics, we don't want \( \theta \), but \( \ln z_{\theta} \) or \( \ln z_{\theta}/z_0 \).
- How do we deal with this in perturbation theory? \( \Rightarrow \) Taylor expansion:

\[ \ln z_{\theta}/z_0 = \ln \left[ \frac{z_{\theta}}{z_0} + O(\theta) \right] \]

= \[ -\frac{3\theta}{4z_0} + \frac{3\theta^2}{8z_0^2} + \ldots \quad \text{(by Mathematica)} \]

- The \( O(\theta) \) term comes from two parts: \( \frac{1}{32 - \theta} \) appearing once and \( -\frac{3\theta}{4z_0} \) appearing twice in the Taylor expansion.
If we look at this in diagrams:

\[
\ln \frac{Z}{Z_0} = 00 + \left\{ 00 \times \frac{00}{00} \right\} + 0 \times 000 - \left\{ 00 \times 00 \right\}
\]

\[
= 00 + \left\{ 0 \right\} + 000 + 0 \times 000
\]

⇒ The "disconnected" parts cancel out when we take the logarithm.

This is just hand waving right now, because we haven’t established that the factors multiplying the cancelling terms are really the same.

But, in fact, the result is true in general and is called

An “linked cluster theorem”.

We can prove it with an elegant technique called

“replica method”, which we will hint at and fill in its
details in Piazza.

Another Teaser: In thermodynamics, we do Legendre transformations (LT)

of thermodynamic potentials. How do we \( LT \ln Z(x) \) and what good is it?
(Hint: we will generate an "effective action".) Later!

Yet another Teaser: Saddle point evaluations of integrals.

Consider \( Z_\lambda \) again but change variables to \( y = \exp x \).

\[
Z_\lambda = \int_{-\infty}^{\infty} \frac{dy}{2\pi i} e^{\lambda (\frac{y^2}{2} + gy)} \Rightarrow \text{look at when } x \text{ gets large and}
\]

The integral is dominated by stationary points

of the integrand (where the derivative of the exponent vanishes).

⇒ a new expansion

\[
Z_\lambda = \exp \left( \frac{(g+\lambda)^2}{2} \right) \left( 1 + \frac{g^2}{2(4\lambda)} \right) \left( 1 + \frac{\lambda}{2(4\lambda)} \right) \left( 1 + \frac{\lambda}{2(4\lambda)} \right) ...
\]

⇒ a much better expansion
Let's carry out the first two orders in $\lambda$ for $\langle \phi^2 \rangle$

\[
\langle \phi^2 \rangle = \left[ \frac{\partial}{\partial \phi} \right]_0 \left[ \frac{\partial}{\partial \phi} \right]_0 \left[ \frac{2 \lambda}{\beta} \phi^2 + \cdots \right]_0 = \left[ \frac{2 \lambda}{\beta} \phi^2 e^{\frac{3}{2} \alpha j} \right]_0
\]

\[
= \left[ \frac{2 \lambda}{\beta} \left[ 1 - \frac{1}{2} \lambda \phi^2 \right] + \cdots \right]_0 \left[ \alpha + \frac{3}{2} \lambda \phi^2 j + \cdots \right]_0
\]

* We have to be careful about expanding consistently because both numerator and denominator.

\[
x_0 : \frac{2 \lambda}{\beta} \left[ 1 - \frac{1}{2} \lambda \phi^2 \right] = \frac{1}{\alpha}
\]

\[
\left[ 1 - \frac{1}{2} \lambda \phi^2 \right] = \frac{1}{\alpha} \quad \text{just end points, not}
\]

This is a "non-interacting propagator." Our Feynman rule is to let the $\phi^2$ represent endpoints on a line.

* Now for $x_0^2 + x_0^4$ order

\[
\frac{1}{\alpha + \frac{2 \lambda}{\beta} \left[ -\frac{1}{2} \lambda \phi^2 \right]} = \frac{1}{\alpha} \left[ \frac{6 \lambda^2}{\beta^2} \left( \frac{4.3}{\phi^4} \right) + O(\lambda^2) \right]
\]

\[
= \left( \frac{1}{\alpha} - \frac{2 \lambda}{\beta} \left( \frac{4.3}{\phi^4} \right) \right) + O(\lambda^2)
\]

\[
= \left( \frac{1}{\alpha} - \frac{2 \lambda}{\beta} \left( \frac{4.3}{\phi^4} \right) \right) + O(\lambda^2)
\]

In diagrams: $\frac{1}{\alpha} \times \frac{1}{4!} \left( \text{the } 4! \text{ from } \langle \phi^2 \rangle \text{ is } 4! \right)$

* We'll find cancellations now between numerator and denominator, which will again leave us with connected diagrams only.
\[ \langle \delta^2 \rangle = \hat{\mathcal{O}} + \mathcal{O}(\lambda^2) + \mathcal{O}(\lambda^3) \]

\[ = \hat{\mathcal{O}} + \mathcal{O}(\lambda^2) \quad \text{(cancel)} \]

\[ \Rightarrow \text{"disconnected" parts cancel. Again, this is general but not completely obvious because of factors.} \]

**Check:** Explain the difference between \( \mathcal{O} \) and \( \hat{\mathcal{O}} \)

- What do our Feynman rules say?
  - We're supposed to get \( \frac{1}{a^2} \frac{1}{\lambda} \) but if we apply the vertex rule to \( \mathcal{O} \), we get \( \lambda(a-b)^2 \).
  - This happens because of an overcounting, which is systematic. We correct with the "symmetry factor".

- Symmetry Factors:
  - We get \( n! \) from Taylor series expansion of \( \lambda \) exponentials, and that almost always cancels the \( n! \) ways of interchanging the vertices.
  - Factoring the factor \( 4! \) from \( \langle \delta^2 \rangle \rangle \) is taken into account in the Feynman rule \( \frac{1}{4!} \).

\[ \Rightarrow \text{The symmetry factor is in correction when these cancellations are incomplete.} \]
We find 3 types of symmetry factors in our diagrams. We identify them by expanding to low orders and observing the patterns.

1. Factor of 5 from each line that starts and ends on the same vertex.

This factor follows by comparing \[ \frac{(S_j^*)^2}{(S_j)^2} \] (1 external to 2 external) lines

\[ \frac{(S_j)^2}{(S_j)^2} \left( \frac{(\sigma_j)}{(\sigma_j)} \right)^4 \left( \frac{(\sigma_j^*)}{(\sigma_j)} \right)^4 = \frac{64}{15} \] (4 pt. function)

\[ \frac{(S_j^*)^2}{(S_j)^2} \left( \frac{(\sigma_j)}{(\sigma_j)} \right)^4 \left( \frac{(\sigma_j^*)}{(\sigma_j)} \right)^4 = -\frac{32}{63} \] (2nd diagram)

vs.

\[ \frac{(S_j)^2}{(S_j)^2} \left( \frac{(\sigma_j)}{(\sigma_j)} \right)^4 \left( \frac{(\sigma_j^*)}{(\sigma_j)} \right)^4 = -\frac{32}{63} \] (3rd diagram)

We'll write these factors as fractions but most often they are collected first as a factor $S$, and then the diagram is multiplied by $S$.

This factor goes away for lines for which the ends are different. This includes fermions.

2. Factor of $\frac{1}{n!}$ for each set of $n$ "equivalent lines", which are lines that run between two different vertices.

\[ \begin{array}{c}
  \circ \quad 2! \quad \circ \\
  \text{ but} \quad \bigcirc 1 \text{ odd} \quad \bigcirc 1 \text{ even}
\end{array} \]

Here $\to$ means the ends are different.

3. Factor of $\frac{1}{P}$ for permutations $P$ of the vertices that leave the diagram unchanged (including any arrows).

external points stay fixed when considering permutations.

Let's do $<S^2>$ at $O(\epsilon)$:

\[ \langle S^2 \rangle = \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 \]

(Disconnected)

\[ \frac{1}{1 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2} \]

Let's do $<S^2>$ at $O(\epsilon)$ again:

\[ \frac{1}{1 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2 + \epsilon^2} \]

(Disconnected)
Fill in the symmetry factors:

1. lines to same vertex
2. equivalent lines
3. vertex permutations

What if there were arrows on the lines? 1) is always 1, 2) only counts for lines in the same direction, yes, no. 3) The permutation cannot change the flow of the lines, so $\not\equiv \not\equiv$ instead of $\equiv$.

If the potential is not contracted to a point, we:

$x \rightarrow y$ where $x, y, \beta, \delta$ are spin indices. Then we lose 2) and have only 3).

For $n$-point functions (as opposed to energy diagrams), the remaining symmetry factor is at most $\sqrt{2}$ and for many cases there are no symmetry factors.

What is an example of a contribution to $\langle \delta^2 \rangle$ that has a vertex permutation factor?
Let's carry out the calculation of $\ln \frac{\tau}{\tau_0}$ at $\alpha(\xi)$:

$$\frac{\tau}{\tau_0} = \frac{100}{\xi^2} \cdot 2^n \quad \text{with} \quad 2^n = \left( \frac{-1}{\alpha} \right) \left( \frac{\alpha}{\xi^2} \right)^n$$

The exact answer is

$$x^2 = 4 \frac{\alpha^2}{a^2} \quad x^2 = \frac{105}{3\alpha^2} \quad x^2 = \frac{3165}{128\alpha^2}$$

**Mathematics:**

$$\ln \frac{\tau}{\tau_0} = -\frac{3\xi^2}{4a^2} + \frac{3\xi^2}{a^2} - \frac{9\xi^2}{4a^2} + O(\xi^4)$$

**Rules:**

1. $e^{-\frac{\alpha}{\xi}} = \frac{1}{\xi^2} \quad \text{as} \quad -\frac{\alpha}{\xi} > -6$ 
2. $\frac{\alpha}{\xi} = \frac{1}{6} \quad \text{symmetry factor}$

$$x^2 = \left( \frac{-3}{4} \right) \frac{\alpha^2}{a^2} \quad \text{and} \quad \frac{\alpha^2}{a^2} = \left( \frac{-3}{4} \right)$$

**Graphs:**

(a) $\bigcirc$  
(b) $\bigotimes$  
(c) $\bigtriangleup$  
(d) $\bigstar$

**Symmetry:**

$$Y_{32} + Y_{34} + Y_{48} + Y_{48}$$

$$\left( \frac{-6}{3} \right) \frac{\alpha^2}{a^2} \quad \text{symmetry} \quad \Rightarrow \frac{3}{a^2} \left( \frac{-6}{a} \right) = -\frac{9\alpha^3}{4a^4}$$
What kind of "partial summations" can we think of?

Examples:

1. For $\mathbb{Z}_{\geq 0}$, sum $\sum_0^\infty + \sum_1^\infty + \sum_2^\infty + \cdots$
   which misses one term at each order

2. For $\mathbb{C}^2$, consider $\sum_0^\infty + \sum_1^\infty + \sum_2^\infty + \cdots$
   How can we sum these?
   Designate the sum with a double line: $\sum_0^\infty + \sum_1^\infty + \sum_2^\infty + \cdots \Rightarrow \sum_0^\infty = (1 - \theta)^{-1}$

   [later: $G = G^0 + G^0 G$]

   The sum is recovered by treating it as an equation:

   $0^\text{th}$: $\sum_0^\infty = \sum_1^\infty + \sum_2^\infty + \cdots$

   $1^\text{st}$: $\sum_0^\infty = \sum_1^\infty + \sum_2^\infty + \sum_3^\infty + \cdots$

   $2^\text{nd}$: $\sum_0^\infty = \sum_1^\infty + \sum_2^\infty + \sum_3^\infty + \sum_4^\infty + \cdots$

   and so on

   More complete summation

   $\sum_0^\infty = \sum_1^\infty + \sum_2^\infty + \cdots$ /

   What are the diagrams?

   $\sum_0^\infty = \sum_1^\infty + \sum_2^\infty + \sum_3^\infty + \sum_4^\infty + \cdots$

   What do you miss? e.g. $\sum_0^\infty$

3. More generally, pick out the "one-particle irreducible" (OPI) pieces: does the diagram fall apart when you cut a line?
   Later: we'll see schemes to sum these.

   The partial summation in 2 can be used to define Hartree-Fock.