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Monday 8805 Class

Model partition function: general form is $Z = \int d\xi \, e^{f(\xi)}$

- Analogies to EFT:
  - At low "energies", we have an expansion of f in the form of a Taylor expansion:

    $$f(\xi) = f(0) + \xi f'(0) + \tfrac{1}{2}\xi^2 f''(0) + \ldots$$

  - Now suppose we know from physics reasons there is a symmetry under $\xi \to -\xi$ (That is Z is unchanged), which will be inherited by the low-energy theory $\Rightarrow$ only even terms in $f(\xi)$.
  - The constant $f(0)$ is additive in $\ln Z$ and doesn't change physics $\Rightarrow$ general expansion to quartic order is

Text

$$Z_\lambda = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi/a}} \, e^{-\frac{a\xi^2}{2} - \frac{\lambda}{4}\xi^4}$$

[Z_λ because approximation: as function of λ will be explored].

- Background on partition functions from 10/22/14 (7)-(8) notes.

- Continue with 10/22/14 (9)-(11)

- Generalized Gaussian result (on Piazza)

$$\int_{-\infty}^{\infty} d\xi_1 \cdots d\xi_n \, e^{-\frac{1}{2}\xi_i A_{ij}\xi_j + j_i\xi_i} = \int_{-\infty}^{\infty} d\xi_1 \cdots d\xi_n \, e^{-\frac{1}{2}\underline{\xi}^T \underline{\underline{A}}\,\underline{\xi} + \underline{j}^T\underline{\xi}}$$

$$= (2\pi)^{n/2} (\det \underline{\underline{A}})^{-1/2} e^{\frac{1}{2} j_i (A^{-1})_{ik} j_k} = (2\pi)^{n/2} (\det \underline{\underline{A}})^{-1/2} e^{\frac{1}{2}\underline{j}^T \underline{\underline{A}}^{-1}\underline{j}}$$

- Let's continue to higher order.

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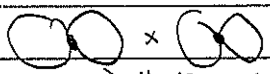
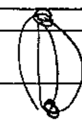
$\mathcal{O}(\lambda^2)$: $\frac{1}{2}\left(\frac{\lambda}{4}\right)^2\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4 \frac{1}{4!}\left(\frac{1}{2}\right)^4 (j a^{-1} j)(j a^{-1} j)(j a^{-1} j)(j a^{-1} j) = \frac{1}{2}\cdot\frac{1}{16}\cdot\frac{1}{4!}\cdot\frac{1}{16}\cdot 8!\,\frac{\lambda^2}{a^4} = \frac{105}{32}\frac{\lambda^2}{a^4}$ ✓

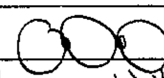
- Many combinatoric factors, but we can organize according to how the $\left(\frac{\partial}{\partial j}\right)^4$ attacks the $j a^{-1} j$ terms, as represented by diagrams:



each $\left(\frac{\partial}{\partial j}\right)^4$ hits two separate $\frac{1}{2} j a^{-1} j$ terms



each $\frac{\partial}{\partial j}$ in $\left(\frac{\partial}{\partial j}\right)^4$ picks up on j from each $\frac{1}{2} j a^{-1} j$ term



for each $\left(\frac{\partial}{\partial j}\right)^4$, two $\frac{\partial}{\partial j}$'s hit one $j a^{-1} j$ term and the other two hit different $j a^{-1} j$'s

note:

is the same!

- Now what if we want an expectation value, such as $\langle \xi^2 \rangle$, which is analogous to a Green's function or correlation function

$$\langle \xi^2 \rangle = \frac{N\int_{-\infty}^{\infty} d\xi\, \xi^2\, e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4}}{N\int_{-\infty}^{\infty} d\xi\, e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4}} = \frac{\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j} Z_\lambda[j]\right)\Big|_{j=0}}{Z_\lambda[j]\big|_{j=0}} \leftarrow \text{first the derivatives, then set } j=0$$

N's cancel

$$= \frac{\left[\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right) e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]_{j=0}}{\left(e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right)_{j=0}}$$

We'll carry this out to a couple of orders, but first let's recall that for thermodynamics we don't want Z_λ, but $\ln Z_\lambda$ or $\ln Z_\lambda / Z_0$.

- How do we do that in perturbation theory? ⇒ Taylor expansion

$$\ln \frac{Z_\lambda}{Z_0} = \ln\left[1 - \frac{3\lambda}{4a^2} + \frac{105}{32}\frac{\lambda^2}{a^4} + \mathcal{O}(\lambda^3)\right]$$

$$= -\frac{3\lambda}{4a^2} + \frac{3\lambda^2}{a^4} + \ldots \quad \text{(by Mathematica)}$$

- The $\mathcal{O}(\lambda^2)$ term comes from two parts: the $\frac{105}{32}\frac{\lambda^2}{a^4}$ appearing once and $-\frac{3\lambda}{4a^2}$ appearing twice in the Taylor expansion.

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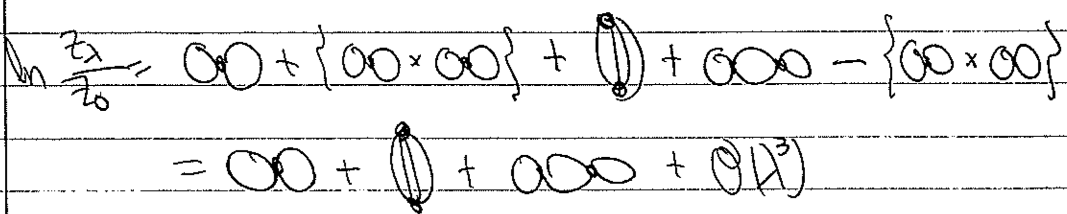
If we look at this in diagrams:



$\Rightarrow$ the "disconnected" parts cancel out when we take the logarithm.

- This is just handwaving right now, because we haven't established that the factors multiplying the cancelling terms are really the same.
- But, in fact, the result is true in general and is called the "linked cluster theorem".
- We can prove it with an elegant technique called the "replica method", which we will hint at and fill in the details in Piazza.

- Another teaser: In thermodynamics, we do Legendre transformations (LT) of thermodynamic potentials. How do we LT $\ln Z_\lambda(j)$ and what good is it? (Hint: we will generate the "effective action") Later!

- Yet another teaser: Saddle point evaluations of integrals.
  Consider Z_λ again but change variables to $y = \sqrt{\lambda}\,\xi$

$$\Rightarrow Z_\lambda = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi\lambda/a}}\, e^{-\frac{1}{\lambda}\left(\frac{ay^2}{2} + \frac{1}{4}y^4\right)}$$

$\Rightarrow$ look at when $\frac{1}{\lambda}$ gets large and the integral is dominated by stationary points of the integrand (where the derivative of the exponent vanishes).

$\Rightarrow$ a new expansion

$$Z_\lambda \doteq \underbrace{\underbrace{\frac{e^{\left(\sqrt{1+4\lambda}-1\right)^2/16\lambda}}{(1+4\lambda)^{1/4}}}_{LO}\left(1 + \frac{12\lambda^2}{2(1+4\lambda)\left(1+\sqrt{1+4\lambda}\right)^2} + \cdots\right)}_{NLO}$$

$\Rightarrow$ a much better expansion!

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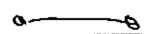
Let's carry out the first two orders in λ for $\langle \xi^2 \rangle$

$$\langle \xi^2 \rangle = \frac{\left[\frac{\partial}{\partial j}\frac{\partial}{\partial j} Z_\lambda[j]\right]\Big|_{j=0}}{Z_\lambda[j]_{j=0}} = \frac{\left\{\frac{\partial}{\partial j}\frac{\partial}{\partial j} e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right\}\Big|_{j=0}}{\left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]\Big|_{j=0}}$$

$$= \frac{\left\{\frac{\partial}{\partial j}\frac{\partial}{\partial j}\left[1 - \frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4 + \dots\right]\left[1 + \frac{1}{2}(j a^{-1} j) + \dots\right]\right\}_{j=0}}{\left\{\left[1 - \frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4 + \dots\right]\left[1 + \frac{1}{2}(j a^{-1} j) + \dots\right]\right\}_{j=0}}$$

- We have to be careful about expanding consistently because of the numerator and denominator:

$$\boxed{\lambda^0 : \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}[1]\left[\frac{1}{2} j a^{-1} j\right]}{[1][1]} = \frac{1}{a}}$$



1/a

just end points; not λ vertices

This is the "non-interacting propagator." Our "Feynman rule" is to let the $\frac{\partial}{\partial j}$'s represent endpoints on a line.

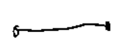
- Now for $\lambda^0 + \lambda^1$ order

$\left(\frac{\partial}{\partial j}\right)^6 (jjjjjj) = 6!$

$$\frac{1}{a} + \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}\left(-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4\right)\frac{1}{3!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)}{1 - \frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4 \frac{1}{2!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)}$$

$$= \left(\frac{1}{a} - \frac{\lambda}{4}\frac{1}{3!}\left(\frac{1}{2}\right)^3 \frac{1}{a^3} 6!\right)\left(1 + \frac{\lambda}{4}\frac{1}{2!}\left(\frac{1}{2}\right)^2 \frac{1}{a^2} 4!\right) + O(\lambda^2)$$

$$= \frac{1}{a} - \frac{\lambda}{a^3}\left[\frac{6\cdot 5\cdot 4}{4\cdot 8} - \frac{4\cdot 3}{4\cdot 4}\right] + O(\lambda^2) = \frac{1}{a} - \frac{3\lambda}{a^3} + O(\lambda^2)$$

- In diagrams  $\frac{1}{a}$  $-\frac{\lambda}{4}4!$ (the 4! is from $\left(\frac{\partial}{\partial j}\right)^4 j^4 = 4!$)

- We'll find cancellations now between numerator and denominator, which will again leave us with connected diagrams only.

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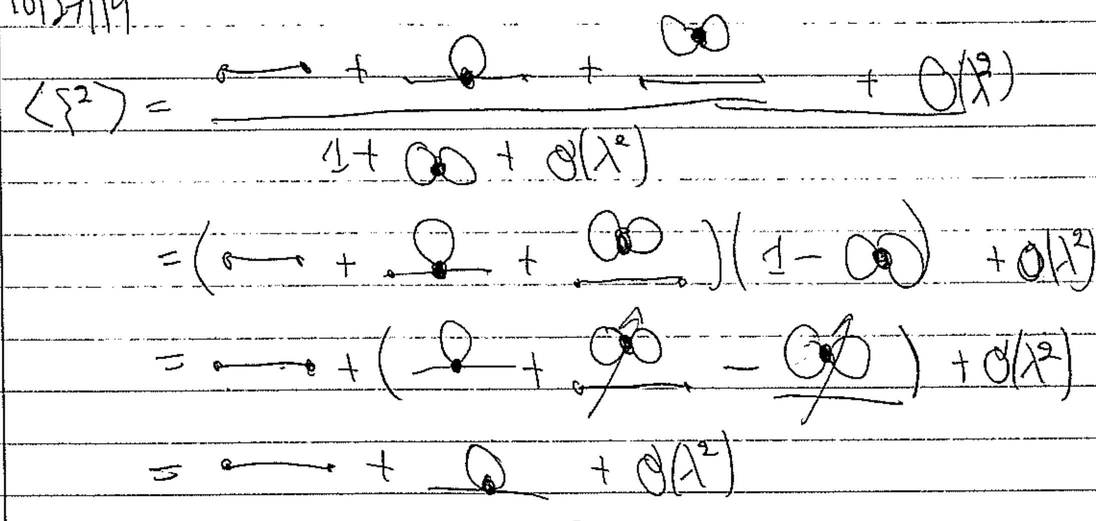

⇒ The "disconnected" parts  cancel. Again, this is general but not completely obvious because of factors.

Check: Explain the difference between  and 

- What do our Feynman rules say?
  - We're supposed to get $\frac{1}{a} - \frac{3\lambda}{a^3}$ but if we apply the vertex rule to  we get $\frac{1}{a} - \frac{6\lambda}{a^3}$.
  - This happens because of an overcounting, which is systematic. We correct with the "symmetry factor".

- Symmetry Factors:
  - We get $\frac{1}{n!}$ from Taylor series expansion of the exponentials, and that almost always cancels the $n!$ ways of interchanging the vertices.
  - Further, the factor $4!$ from $\left(\frac{\partial}{\partial j}\right)^4 j^4$ is taken into account in the Feynman rule $-\frac{\lambda}{4!} \cdot 4!$

⇒ The symmetry factor is the correction when these cancellations are incomplete.

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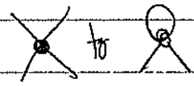
- We find 3 types of symmetry factors in our diagrams. We identify them by expanding to low orders and observing the patterns.

① factor of $\frac{1}{2}$ from each line that starts and ends on the same vertex.

e.g. 

- This factor follows by comparing  (4 external to 2 external lines)

$$\left(\frac{\delta}{\delta_j}\right)^4\left(-\frac{\lambda}{4}\right)\left(\frac{\delta}{\delta_j}\right)^4\frac{1}{4!}\left(\frac{1}{2}\right)^4\left(j\,a^{-1}j\right)^4 = -\frac{6\lambda}{a^4} \quad \text{(4 pt. function)}$$

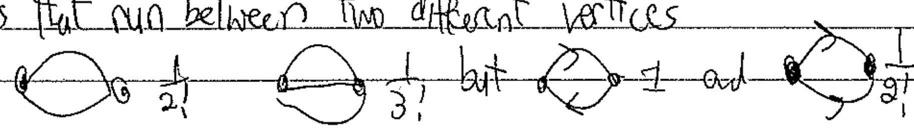
vs.

$$\left(\frac{\delta}{\delta_j}\right)^2\left(-\frac{\lambda}{4}\right)\left(\frac{\delta}{\delta_j}\right)^4\frac{1}{3!}\left(\frac{1}{2}\right)^3\left(j\,a^{-1}j\right)^3 = -\frac{3\lambda}{a^3}$$

$\Rightarrow$ $\frac{1}{2}$ factor for 2nd diagram (or think of opening the loop)

- We'll write these factors as fraction, but most often they are collected first as a factor S, and then the diagram is multiplied by $\frac{1}{S}$.
- This factor goes away for lines for which the ends are different (this includes fermions).

② factor of $\frac{1}{n!}$ for each set of n "equivalent lines", which are lines that run between two different vertices

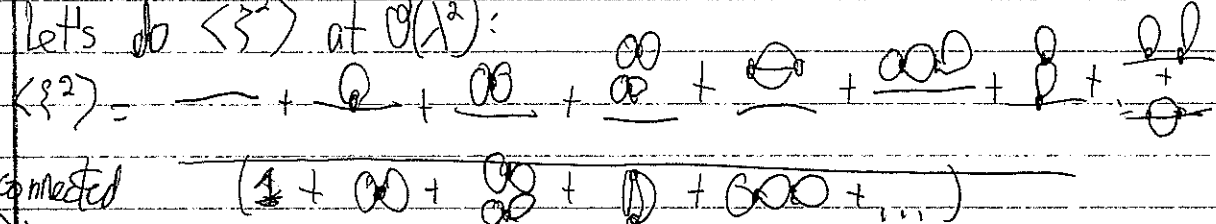


(here → means the ends are different)

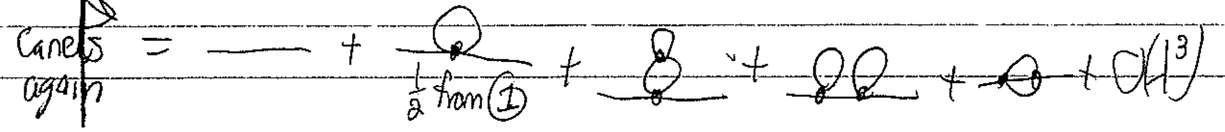
③ Factor of $\frac{1}{P}$ for permutations P of the vertices that leave the diagram unchanged (including any arrows).

- external points stay fixed when considering permutations.

Let's do $\langle \xi^2 \rangle$ at $O(\lambda^2)$:



disconnected cancels again



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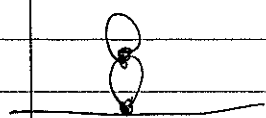
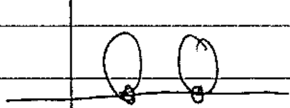
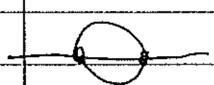
Fill in the symmetry factors



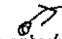
① line to same vertex!
② equivalent lines!
③ vertex permutations!

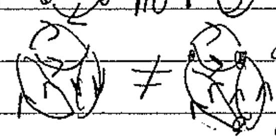


① lines to same vertex:
② equivalent lines!
③ vertex permutations:

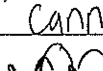


① lines to same vertex
② equivalent lines!
③ vertex permutations!

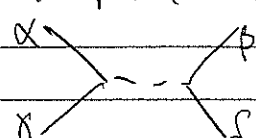
What if there were arrows on the lines? ① is always 1. ② only counts for lines in the same direction.  yes,  no. ③ The permutation cannot change the flow of the lines, so



← 4 in
← 4 out
instead on 2 in and 2 out

 is still $\frac{1}{2}$, but under ②.

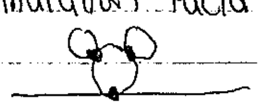
If the potential is not contracted to a point, e.g.:



where $\alpha, \gamma, \beta, \delta$ are spin indices, then we lose ② and have only ③.

For n-point functions (as opposed to energy diagrams), the remaining symmetry factor is at most 1/2 and for many cases there are no symmetry factors.

What is an example of a contribution to $\langle \rho^2 \rangle$ that has a vertex permutation factor?



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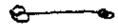
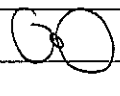
Let's carry out the calculation of $\ln \frac{Z}{Z_0}$ at $O(\lambda^3)$:

$\boxed{\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \lambda^n Z_n}$ with $\boxed{Z_n = \frac{(-1)^n}{n!} \frac{(4n-1)!!}{4^n} \frac{1}{a^{2n}}}$ the exact answer

$\Rightarrow \lambda Z_1 = -\frac{3}{4}\frac{\lambda}{a^2} \qquad \lambda^2 Z_2 = \frac{105}{32}\frac{\lambda^2}{a^4} \qquad \lambda^3 Z_3 = -\frac{3465}{128 a^6}\lambda^3$

Mathematica: $\boxed{\ln \frac{Z}{Z_0} = -\frac{3\lambda}{4a^2} + \frac{3\lambda^2}{a^4} - \frac{99\lambda^3}{4a^6} + O(\lambda^4)}$

rules:  $\frac{1}{a}$  $-\frac{\lambda}{4} 4! = -6\lambda$ ①, ②, ③ symmetry factors

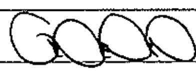
λ^1:  $\left(-\frac{\lambda}{4}4!\right)\frac{1}{a^2}$ ① $\frac{1}{2}\cdot\frac{1}{2}$ ② $1/2$ ③ 1 $\Rightarrow \frac{1}{a^2}(-6\lambda)\frac{1}{8} = -\frac{3\lambda}{4a^2}$ ✓

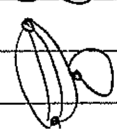
λ^2: a)  +  b)

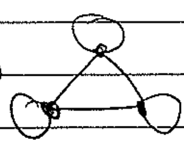
	a)	b)
①	1	$\frac{1}{2}\cdot\frac{1}{2}$
②	$\frac{1}{4!}$	$1/2$
③	$\frac{1}{2}$	$1/2$

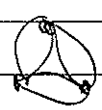
$\Rightarrow (-6\lambda)^2 \frac{1}{a^4}\left(\frac{1}{48} + \frac{1}{16}\right) = \frac{+3\lambda^2}{a^4}$ ✓

λ^3:

a) 

b) 

c) 

d) 

	a)	b)	c)	d)
①	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2!}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	1
②	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{3!}$	1	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$
③	$\frac{1}{2}$	$\frac{1}{2!}$	$\frac{1}{3!}$	$1/3!$
	$1/32$ +	$1/24$ +	$1/48$ +	$1/48$

$(-6\lambda)^3 \frac{1}{a^6}$ × symmetry $\Rightarrow -\frac{\lambda^3}{a^6} 6^3 \left(\frac{11}{96}\right) = -\frac{99\lambda^3}{4a^6}$ ✓

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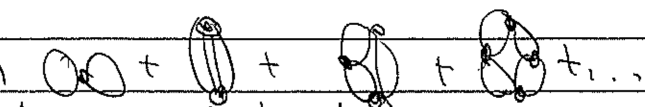
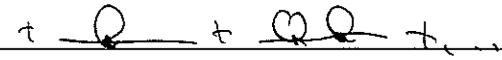
What kind of "partial summations" can we think of?

Examples:

① For $\ln Z/Z_0$, sum  which picks one term at each order

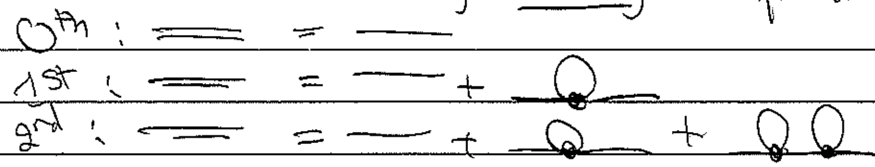
② For $\langle \xi^2 \rangle$, consider 

How can we sum these?

Designate the sum with a double line: ═══, then

[Later: $G = G^0 + G^0 \Sigma G$]

- The sum is recovered by iterating the equation:



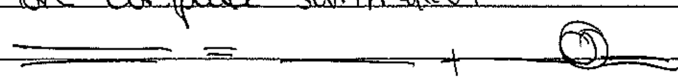
0th: 1st term

1st: use 0th for ═ on right side

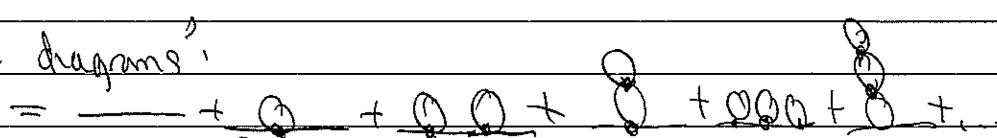
2nd: use 1st for ═ on right side

and so on

More complete summation



What are the diagrams?



What do you miss? e.g. 

③ More generally, pick out the "one particle irreducible" (1PI) pieces: does the diagram fall apart when you cut a line?

- Later: we'll see schemes to sum these.

- The partial summation in ② can be used to define Hartree-Fock.