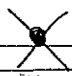
①

10/29/14

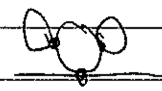
Wednesday 8805 Class

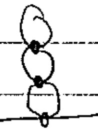
• Recap: Symmetry factor rules ① $\frac{1}{2}$ for each (line to same vertex)
② $\frac{1}{n!}$ for each n-equivalent lines $\frac{1}{3!}$
③ $\frac{1}{P}$ for vertex permutation P

Other Feynman rules: —— $\frac{1}{a}$

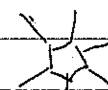
 $-\frac{\lambda}{4!}4! = -6\lambda$

With arrows: ① ⇒ 1 always ② $=1$, $=\frac{1}{2!}$
③ permutation must preserve lines

Feynman rule check for $\langle \xi^2 \rangle$ (a)  $\frac{1}{2^2} \times 1 \times \frac{1}{2} \times (-6\lambda)^3 \frac{1}{a^7} = \frac{27\lambda^3}{a^7}$

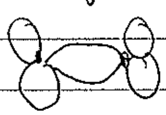
Piazza problem: find $\langle \xi^2 \rangle$ to 3rd order. (b)  $\frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times 1 \times (-6\lambda)^3 \frac{1}{a^7} = \frac{27\lambda^3}{a^7}$

Now suppose $Z = N\int_{-\infty}^{\infty} d\xi \, e^{-(a\xi^2/2 + \alpha\xi^6/6)}$ ← rules determined by "Lagrangian"

What are the Feynman rules? —— $\frac{1}{a}$  $-\frac{\alpha}{6}6!$ ← from $\left(\frac{d}{dJ}\right)^6 J^6 = 6!$

from overall "−"

from factor in $\mathcal{L}$

Piazza: $\frac{Z_\alpha}{Z_0}$ diagram:  $\Rightarrow \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2!}\right)^3 \times \frac{1}{2} \times \left(-\frac{\alpha}{6}6!\right)\frac{1}{a^6} = \frac{225}{4}\frac{\alpha^2}{a^6}$

• Recap the features of Feynman rules from ⑥ on 10/22/14
  - connected diagrams only → do replica method on ②, ③
  - lines are propagators ⇒ from Gaussian integral ⇒ inverse of quadratic term: $\xi^T a \xi \rightarrow$ •——• a^{-1}
  - vertices have rules that come directly from the "Lagrangian" interaction
  - symmetry factor adjust the combinatoric factors
  - diagrams for energy are closed; those for expectation values are open.
  - Sums over any intermediate indices (coordinates)
    ⇒ see the simple example on ④
  - * Now: for non-perturbative summations (infinite subsets of diagrams) we can devise diagrammatic equation (Dyson equations)
    ⇒ pg. ⑨ for 10/27/14.

What about $\langle \xi^4 \rangle$? ...lly connected ...agrams for $\langle \xi^4 \rangle - 3\langle \xi^2 \rangle \langle \xi^2 \rangle$

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<u>Replica method</u>: The idea is to consider n identical copies of the partition function multiplied together: $\tilde{Z}^n \equiv \left(\frac{Z_J}{Z_0}\right)^n$.
Now rewrite $\tilde{Z}^n$ so it takes the form of a series expansion in n:

$$\boxed{\tilde{Z}^n = e^{\ln \tilde{Z}^n} = e^{n \ln \tilde{Z}} = 1 + n(\ln \tilde{Z}) + \frac{1}{2} n^2 (\ln \tilde{Z})^2 + \dots}$$

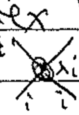
So to find $\ln \tilde{Z}$, we calculate $\tilde{Z}^n$ and then identify the linear term in n.

But $\tilde{Z}^n$ is just a product of n $\tilde{Z}$'s, with variables $\xi_1, \xi_2, \dots, \xi_n$ and corresponding sources:

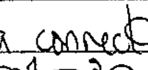
$$\boxed{\begin{aligned}\tilde{Z}^n &= \left(\int d\xi_1\, e^{-\frac{1}{2} a_1 \xi_1^2 - \frac{\lambda_1}{4}\xi_1^4 + J_1 \xi_1}\right)\left(\int d\xi_2\, e^{-\frac{1}{2} a_2 \xi_2^2 - \frac{\lambda_2}{4}\xi_2^4 + J_2 \xi_2}\right) \cdots \left(\int d\xi_n\, e^{\cdots}\right) \\ &= \int d\xi_1 \cdots d\xi_n\, e^{-\frac{1}{2}\sum_{i=1}^{n} a_i \xi_i^2 - \frac{1}{4}\sum_i \lambda_i \xi_i^4 + \sum_i J_i \xi_i}\end{aligned}}$$

- Even though the copies are identical, we add indices to a and λ.

- Now use $\frac{\partial}{\partial J_i}$ as before to remove interaction terms from <u>all</u> of the integrals.

* • A vertex has the same index i for all lines coming out of it , so the $1/a_i$'s have the same i index at each end.

- we also sum over i from 1 to n.

* • It follows that a connected diagram can only have vertices of one i at a time. Eg.  can't happen, because the 1 vertex comes from $-\frac{\lambda_1}{4}\left(\frac{\partial}{\partial J_1}\right)^4$ and each $\frac{\partial}{\partial J_1}$ is only nonzero if acting on $\frac{1}{2} J_1 a_1^{-1} J_1$. So the lines from "1" to "2" must be "1" lines. But the same argument applied to the other vertex says they must be "2" lines, so the diagram cannot occur.

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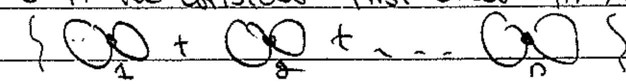
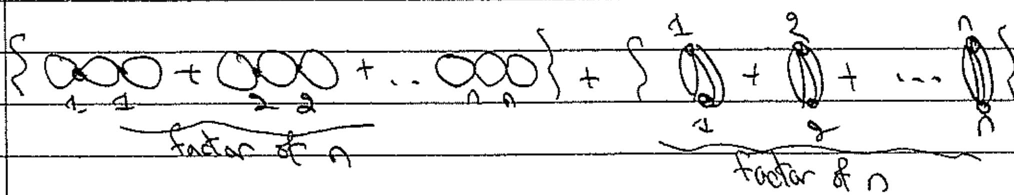
- So if we consider first order in λ:

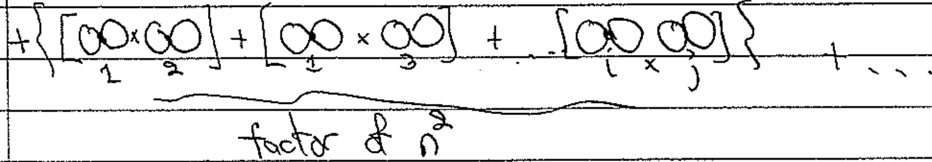


$\Rightarrow$ overall factor of n

[all diagrams have the same value]

- At 2^{nd} order in λ:





- So if we have 2 disconnected pieces, the total is $\propto n^2$, three disconnected pieces the total is $\propto n^3$, and so on.

$\Rightarrow$ The terms linear in n are precisely those that are connected. But that is also $\ln \tilde{Z}$

$\Rightarrow$ $\ln Z - \ln Z_0$ is exactly given by the sum of connected diagrams, with all of the correct factors!

(because these are all the contributions)

QED.

- The replica argument can be used to show that $\langle \xi^2 \rangle$ is only from connected diagrams with two hanging lines. [see Piazza] That is, diagrams of the form  only contribute.

connected

* - It follows similarly that the expectation value $\langle \hat{O} \rangle$ for any operator $\hat{O}$ is given by the sum of connected diagrams.

Aside: Generalized model partition function (repeated indices summed 1 to n)

$$Z_\lambda[j_i] = N\int_{-\infty}^{\infty} d\xi_1 \cdots d\xi_n \, e^{-\frac{1}{2}\xi_i A_{ik}\xi_k - \frac{1}{4}\lambda\sum_i \xi_i^4 + j_i\xi_i}$$

← e.g. $j_1\xi_1 + j_2\xi_2 + \ldots$ (like integrations and spin sums)

$$= N' e^{-\frac{1}{4}\lambda\sum_i\left(\frac{\partial}{\partial j_i}\right)^4} e^{\frac{1}{2}j_i(A^{-1})_{ik}j_k}$$

$\boxed{N' \text{ is independent of } \lambda}$

↑ Green's function (propagator)

vertex all at the same "x" point, then integrated over all x
$-\frac{1}{4}\lambda\left(\xi_1^4 + \xi_2^4 + \ldots\right)$

So suppose $n=2$. Then

" $A_{ik} \neq 0$ for $i \neq k$ if there are derivatives"

$$\frac{Z_\lambda}{Z_0} = e^{-\frac{\lambda}{4}\left(\left(\frac{\partial}{\partial j_1}\right)^4 + \left(\frac{\partial}{\partial j_2}\right)^4\right)} e^{\frac{1}{2}\left[j_1 A^{-1}_{11} j_1 + j_1 A^{-1}_{12} j_2 + j_2 A^{-1}_{21} j_1 + j_2 A^{-1}_{22} j_2\right]}$$

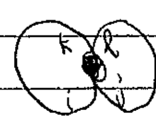
$\boxed{A^{-1}_{11} \equiv (A^{-1})_{11}}$

Order λ result:

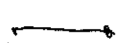
$$-\frac{\lambda}{4}\left(\left(\frac{\partial}{\partial j_1}\right)^4 + \left(\frac{\partial}{\partial j_2}\right)^4\right)\frac{1}{2!}\left(\frac{1}{2}\right)^2\left(j_1 A^{-1}_{11} j_1 + j_1 A^{-1}_{12} j_2 + j_2 A^{-1}_{21} j_1 + j_2 A^{-1}_{22} j_2\right)^2$$

$$= -\frac{\lambda}{4}\cdot\frac{1}{8}\left(A^{-1}_{11}\cdot A^{-1}_{11} + A^{-1}_{22}A^{-1}_{22}\right)\cdot 4! = -\frac{3}{4}\lambda\left(A^{-1}_{11} + A^{-1}_{22}\right)$$

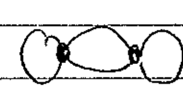
↑ same symmetry factor!

So the diagram is still 

and  is $-\frac{\lambda}{4}\cdot 4!$ with $\delta_{k\ell}\delta_{ij}\delta_{ik}$ (all the same index)

and  $\Rightarrow (A^{-1})_{ik}$ ← new rule: assign indices to the lines and vertices

And we sum over repeated indices!

Does $(A_{12})^{-1}$ ever appear? [Hint: consider ]

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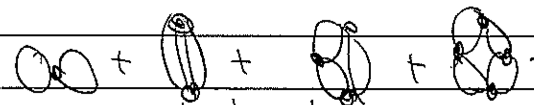
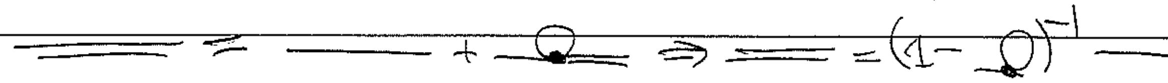
What kind of "partial summations" can we think of?

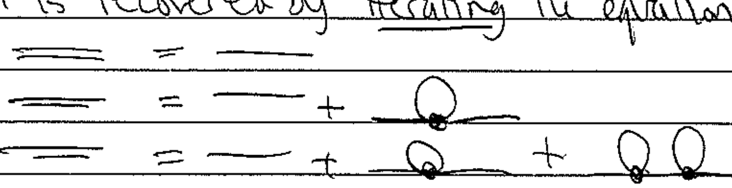
Examples:

① For $\ln Z/Z_0$, sum  + ...
which picks one term at each order

② For $\langle \xi^2 \rangle$, consider  + ...
How can we sum these?
Designate the sum with a double line: ═══, then



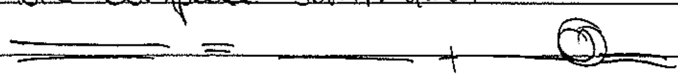
[Later: $G = G^0 + G^0 \Sigma G$]
- The sum is recovered by iterating the equation:



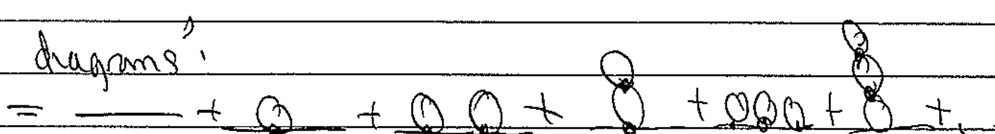
0th: 1st term
1st: use 0th for ═ on right side
2nd: ← use 1st for ═ on right side

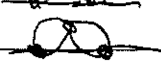
and so on

More complete summation



What are the diagrams?



What do you miss? e.g. 

③ More generally, pick out the "one particle irreducible" (1PI) pieces: does the diagram fall apart when you cut a line?
- Later: we'll see schemes to sum these.

- The partial summation in ② can be used to define Hartree-Fock.

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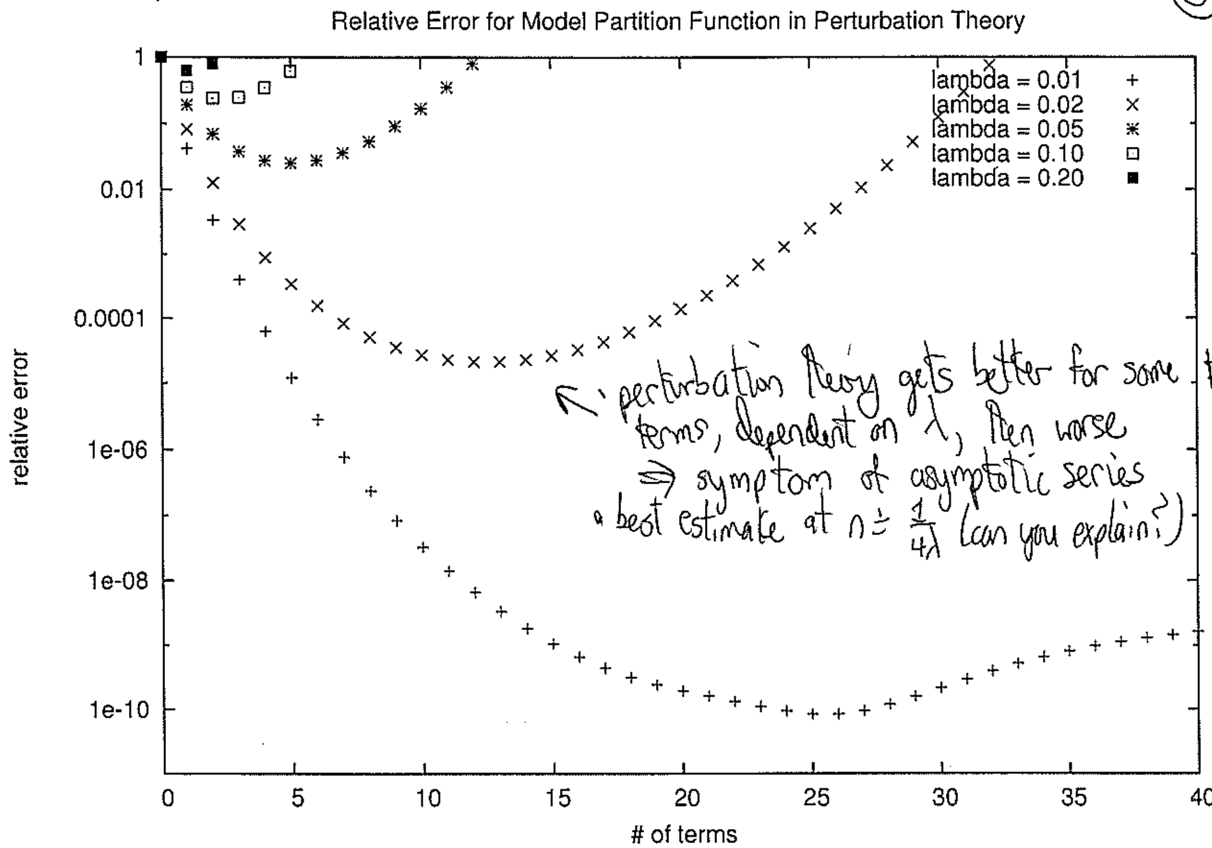
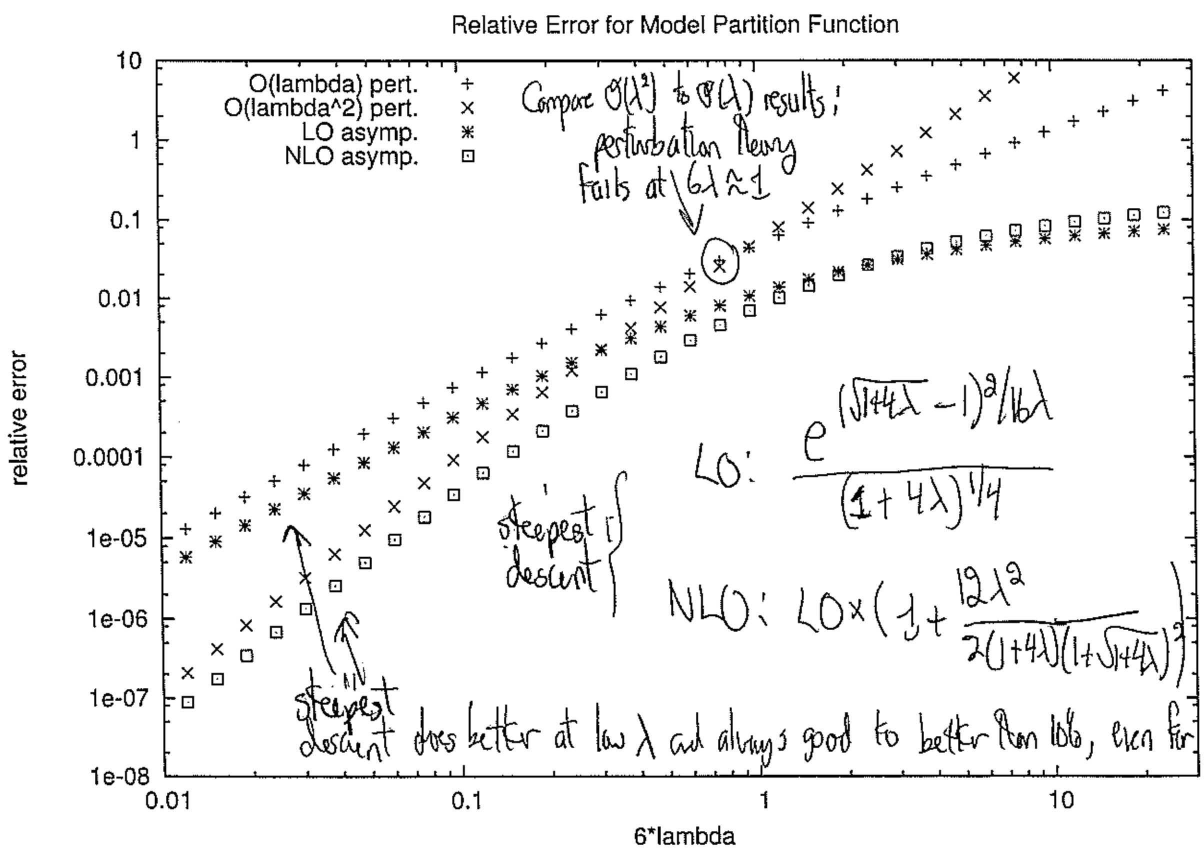


← perturbation theory gets better for some # of terms, dependent on λ, then worse
⇒ symptom of asymptotic series
• best estimate at $n \doteq \frac{1}{4\lambda}$ (can you explain?)



Compare $\mathcal{O}(\lambda^2)$ to $\mathcal{O}(\lambda)$ results; perturbation theory fails at $6\lambda \simeq 1$

$$LO: \frac{e^{(\sqrt{1+4\lambda}-1)^2/16\lambda}}{(1+4\lambda)^{1/4}}$$

steepest descent

$$NLO: LO \times \left(1 + \frac{12\lambda^2}{2(1+4\lambda)(1+\sqrt{1+4\lambda})^2}\right)$$

steepest descent does better at low λ and always good to better than 10%, even for large λ

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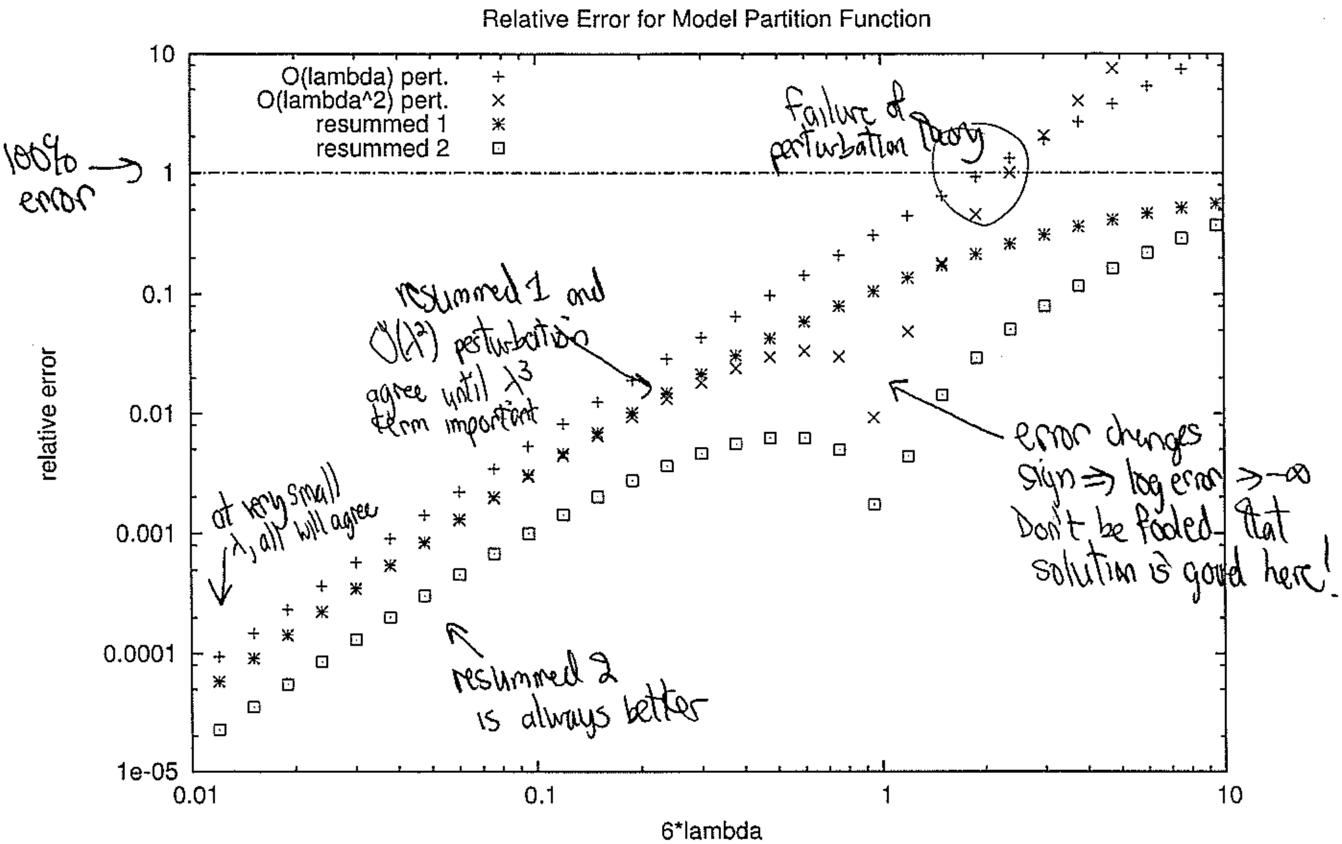
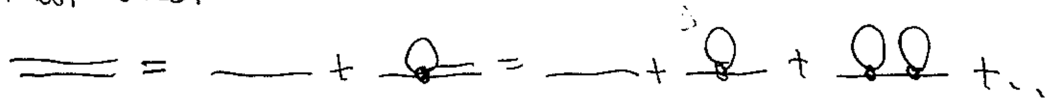


- Let's consider two partial resummations of the perturbation series for $\langle \xi^2 \rangle$ by looking at the diagrams and applying the Feynman rules.

- First try

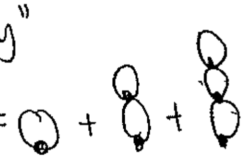


$\bigcirc \Rightarrow -\frac{6\lambda}{a}\cdot\frac{1}{2} \Rightarrow \langle \xi^2 \rangle \doteq \frac{1}{a} + \frac{1}{a}\left(-\frac{3\lambda}{a}\right)\frac{1}{a} + \frac{1}{a}\left(-\frac{3\lambda}{a}\right)\frac{1}{a}\left(-\frac{3\lambda}{a}\right)\frac{1}{a} + \dots = \frac{1}{a}\left(1+\frac{3\lambda}{a^2}\right)^{-1}$

call this "resummed 1"

"self-energy"

- Now suppose we want to sum: $\Sigma =$



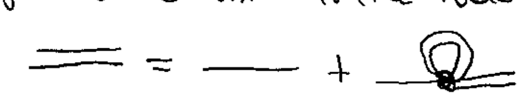
$= -\frac{3\lambda}{a} + \frac{1}{a^3}(-6\lambda)^2\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{a^5}(-6\lambda)^3\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}$

$= -\frac{3\lambda}{a}\left(1+\frac{3\lambda}{a^2}\right)^{-1}$

$\Rightarrow$ resummed 2: $\langle \xi^2 \rangle \doteq \frac{1}{a} + \frac{1}{a}\left(-\frac{3\lambda}{a}\left(1+\frac{3\lambda}{a^2}\right)^{-1}\right)\frac{1}{a} + \dots = \frac{1}{a}\left(1+\frac{3\lambda}{a^2}\left(1+3\frac{\lambda}{a^2}\right)^{-1}\right)^{-1}$

- Can you do the full "Hartree-Fock summation?

$\Rightarrow$



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Without deriving the path integral version of the partition function, let's jump right to the comparison expressions

Feature	Model Z_λ	Pionless EFT with C_0
partition function	$Z_\lambda = N\int d\xi\, e^{-\frac{a\xi^2}{2}-\frac{\lambda\xi^4}{4}}$	$Z=\int \mathcal{D}[\psi^\dagger(x)\psi(x)]\, e^{-\int_0^\beta d\tau \int d^3x\, \psi^\dagger_\alpha(x)\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\psi_\alpha(x)}$ $[x\equiv\{\vec{x},\tau\}]\quad \times e^{-\frac{C_0}{2}\int_0^\beta d\tau\int d^3x\, \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)}$
action (Euclidean)	$Z_\lambda = N\int d\xi\, e^{-S_E}$ $S_E=\frac{a\xi^2}{2}+\frac{\lambda\xi^4}{4}$	$Z=\int\mathcal{D}[\psi^\dagger(x)\psi(x)]\, e^{-S_E}$ $S_E=\int_0^\beta d\tau\int d^3x\, \psi^\dagger_\alpha(x)\underbrace{\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)}_{\equiv \mathcal{G}_0^{-1}}\psi_\alpha(x)+\frac{C_0}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)$
add sources	$Z_\lambda[j]=N\int d\xi\, e^{-S_E+j\xi}$	$Z[\eta_\alpha,\eta^\dagger_\beta]=\int\mathcal{D}[\psi^\dagger\psi]\, e^{-S_E+\int_0^\beta d\tau\int d^3x\, \eta^\dagger_\alpha(x)\psi_\alpha(x)+\psi^\dagger_\alpha(x)\eta_\alpha(x)}$
source rule	$\frac{\partial}{\partial j}j=1 \Rightarrow \frac{\partial}{\partial j_\alpha}j_\beta=\delta_{\alpha\beta}$ $\frac{\partial}{\partial j}\int d\xi\, e^{j\xi}=\int d\xi\,\xi\, e^{j\xi}$	$\frac{\delta}{\delta\eta_\alpha(x)}e^{-\int\psi^\dagger_\beta\eta_\beta}=e^{-\int\psi^\dagger_\beta\eta_\beta}\,\psi^\dagger_\alpha(x)$ $-\frac{\delta}{\delta\eta^\dagger_\alpha(x)}e^{-\int\eta^\dagger_\beta\psi_\beta}=e^{-\int\eta^\dagger_\beta\psi_\beta}\,\psi_\alpha(x)$ Grassman: $\eta_\alpha\eta_\beta=-\eta_\beta\eta_\alpha$, $\eta^\dagger_\beta\psi_\alpha=-\psi_\alpha\eta^\dagger_\beta$ for all pairs η,ψ
remove interaction	$Z_\lambda[j]=Ne^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}$ $\times\int_{-\infty}^{\infty}d\xi\, e^{-\frac{a}{2}\xi^2+j\xi}$	$Z[\eta_\alpha,\eta^\dagger_\beta]=e^{-\int_0^\beta d\tau\int d^3x\,\frac{C_0}{2}\left(\frac{\delta}{\delta\eta_\alpha(x)}\frac{\delta}{\delta\eta_\beta(x)}\frac{-\delta}{\delta\eta^\dagger_\beta(x)}\frac{-\delta}{\delta\eta^\dagger_\alpha(x)}\right)}$ $\times\int\mathcal{D}[\psi^\dagger\psi]\, e^{-\int_0^\beta d\tau\int d^3x\left[\psi^\dagger_\alpha(x)\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\psi_\alpha(x)+\eta^\dagger_\alpha(x)\psi_\alpha(x)+\psi^\dagger_\alpha(x)\eta_\alpha(x)\right]}$
complete the square	$N\int_{-\infty}^{\infty}d\xi\, e^{-\frac{a}{2}\left(\xi+\frac{j}{a}\right)^2}e^{\frac{a}{2}\frac{j^2}{a^2}}$ $=Z_0\, e^{\frac{1}{2}ja^{-1}j}$ after $\xi'=\xi+\frac{j}{a}$	$\int\mathcal{D}(\psi^\dagger\psi)\, e^{-\int\psi^\dagger_\alpha\mathcal{G}^{0-1}_{\alpha\beta}\psi_\beta+\eta^\dagger_\alpha\psi_\alpha+\psi^\dagger_\alpha\eta_\alpha}$ (indices for spin and space-time) $=\int\mathcal{D}(\psi^\dagger\psi)\, e^{-(\psi^\dagger_\alpha+\eta^\dagger_\gamma\mathcal{G}^0_{\gamma\alpha})\mathcal{G}^{0-1}_{\alpha\beta}(\psi_\beta+\mathcal{G}^0_{\beta\delta}\eta_\delta)}$ $\times e^{\int\eta^\dagger_\gamma\mathcal{G}^0_{\gamma\delta}\eta_\delta}$ $=e^{\eta^\dagger_\gamma\mathcal{G}^0_{\gamma\delta}\eta_\delta}\int\mathcal{D}[\psi^\dagger\psi]\, e^{-\int\psi'^\dagger_\alpha\mathcal{G}^0_{\alpha\beta}\psi'_\beta}=e^{\int\eta^\dagger_\gamma\mathcal{G}^0_{\gamma\delta}\eta_\delta}Z_0$
propagator	—— $1/a$	$\vec{x}'\tau'$, β —— $\vec{x}\tau$, α : $\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau,\vec{x}'\tau')$
vertex	$-\frac{\lambda}{4}4!$	$\gamma,\delta,\alpha,\beta$: $-C_0$ ($\delta_{\alpha\gamma}\delta_{\beta\delta}$ + $\delta_{\alpha\delta}\delta_{\beta\gamma}$) and -1 for $\delta_{\alpha\alpha}$ from $-$ sign in exponent

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What is $\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau, \vec{x}'\tau')$?

- There are various ways you can derive it

Def'n: $\mathcal{G}^0_{\alpha\beta}$ is the solution to $\mathcal{G}^{0-1}\mathcal{G}^0 = 1$, which becomes

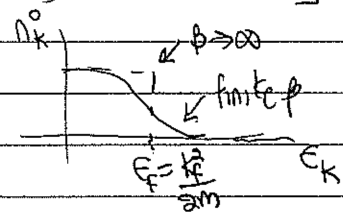
$$\delta_{\alpha\beta}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau; \vec{x}'\tau') = \delta_{\alpha\gamma}\delta^3(\vec{x}-\vec{x}')\delta(\tau-\tau')$$

with the finite temperature boundary condition $\mathcal{G}^0_{\alpha\beta}(\vec{x}0; \vec{x}'\tau') = -\mathcal{G}_{\alpha\beta}(\vec{x}\beta; \vec{x}'\tau')$

① Guess the answer and check that it works

$$\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau, \vec{x}'\tau') = \delta_{\beta\gamma}\frac{1}{\Omega}\sum_k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-(\epsilon^0_k-\mu)(\tau-\tau')}$$
$$\times\left[\theta(\tau-\tau'-\eta)(1-n^0_k) - \theta(\tau'-\tau+\eta)\, n^0_k\right]$$

where $n^0_k = \dfrac{1}{e^{\beta(\epsilon_k-\mu)}+1} \xrightarrow[\beta\to\infty]{} \Theta(\mu-\epsilon_k)$



- And $\frac{1}{\Omega}\sum_k \xrightarrow[\Omega\to\infty]{} \int\frac{d^3k}{(2\pi)^3}$

- Satisfies BC and differential equation (Piazza!)
  - remember $\frac{\partial}{\partial\tau}\theta(\tau-\tau') = \delta(\tau-\tau')$ and $\frac{\partial}{\partial\tau}\theta(\tau'-\tau) = -\delta(\tau-\tau')$

② Solve the differential equation explicitly by using a Fourier transform from $\vec{x}$ to $\vec{k}$ and then doing "division of regions" method. (Piazza!)

③ With field operators (e.g. $\hat{\psi}(\vec{x}) = \frac{1}{\sqrt{\Omega}}\sum_{\vec{k}\lambda} e^{i\vec{k}\cdot\vec{x}}\eta_\lambda a_{\vec{k}\lambda} \to$ more later)

$$\mathcal{G}_{\alpha\beta}(\vec{x}\tau, \vec{x}'\tau') = \frac{\int\mathcal{D}[\psi^\dagger\psi]\,\psi_\alpha(\vec{x}\tau)\psi^\dagger_\beta(\vec{x}'\tau')\, e^{-S_E}}{\int\mathcal{D}[\psi^\dagger\psi]\, e^{-S_E}} \xrightarrow[H\to H_0]{} \mathcal{G}^0_{\alpha\beta}$$

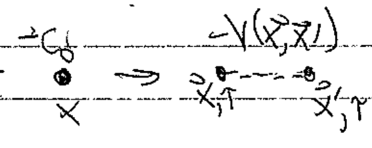
$$= \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}T_\tau\left[\hat{\psi}_\alpha(\vec{x}\tau)\hat{\psi}^\dagger_\beta(\vec{x}'\tau')\right]\right] / \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]$$

} follow up in extra notes

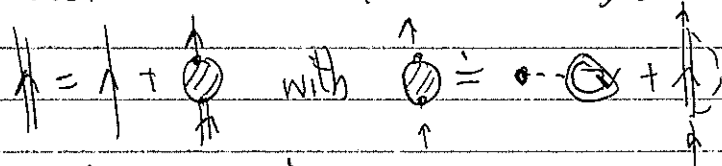
10/29/14

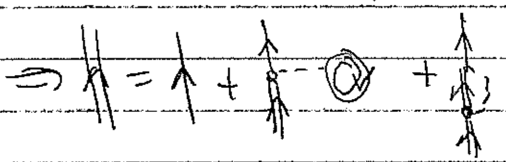
What if we have a finite range (spin-independent) potential?

Then
$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$$
$$= \int \mathcal{D}[\psi^\dagger(\vec{x},\tau)\psi(\vec{x},\tau)]\, e^{-\int_0^\beta d\tau \int d^3x\, \psi^\dagger_\alpha(\vec{x},\tau)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\psi_\alpha(\vec{x},\tau)}$$
$$\times\, e^{-\frac{1}{2}\int_0^\beta d\tau \int d^3x \int d^3x'\, \psi^\dagger_\beta(\vec{x},\tau)\psi^\dagger_\alpha(\vec{x}',\tau) V(\vec{x}-\vec{x}')\psi_\alpha(\vec{x}',\tau)\psi_\beta(\vec{x},\tau)}$$

- Feynman rules go through except 

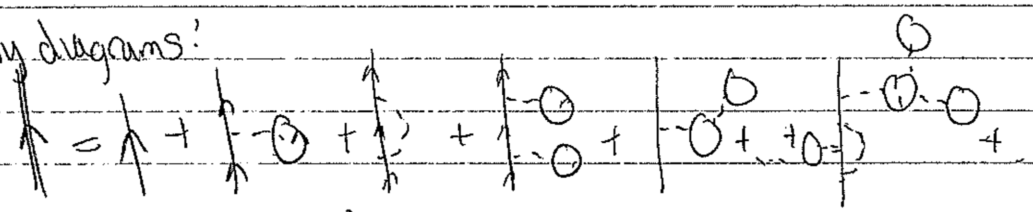
- How do we recover Hartree-Fock for a finite system?

Claim: solve 



self-consistent! ⇑ depends on ⊘ and ⊘ depends on ⇑

Many diagrams:



- What is not included?

- Next time! How we get the HF single-particle wavefunctions and S-equation.