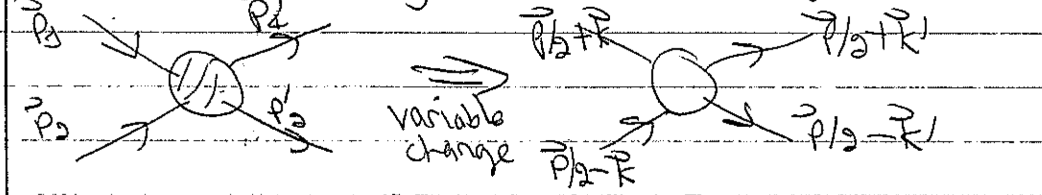
3. Scattering Theory 1

- A more complete version of these notes is available on the 8805.01 website.

- Overview
  - Main source of info on NN force is NN scattering
  - You've seen at least the basics of scattering in QM classes
    $\Rightarrow$ here: review and extend (also in exercises)

  - Neglect V_{em} and n,p mass difference $\Rightarrow m \equiv \frac{1}{2}(m_n + m_p)$
    $\Rightarrow$ generic scattering of two equal-mass particles (nonrelativistic)
    - interacting with a short-ranged potential (does this mean zero or dying off fast at large distances?)

assumes definite momentum



$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

"relative" momentum $\vec{k} = \dfrac{\vec{p}_1 - \vec{p}_2}{2}$ "total" momentum $\vec{P} = \vec{p}_1 + \vec{p}_2$

other conventions exist for relative and center-of-mass coordinates.

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \hat{V} \longrightarrow \hat{T}_{cm} + \hat{H}_{rel} = \frac{\hat{P}^2}{2M} + \frac{\hat{k}^2}{2\mu} + \hat{V}$$

com; "intrinsic" or "relative"; key: independent of COM

$\boxed{M = m_1 + m_2 = 2m}$; $\boxed{\mu = \dfrac{m_1 m_2}{M} = \dfrac{m}{2}}$

- So $|\Psi\rangle = |P\rangle|\psi_{rel}\rangle$ $\leftarrow$ all the physics! Ignore $|P\rangle \Rightarrow$ or in com frame

  $\leftarrow$ drop the rel

  plane wave eigenstate of $\hat{p}$

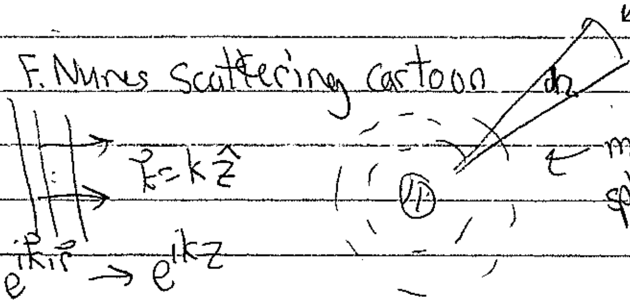
(origin of term?)

"on-shell" means $E_2 = \dfrac{p_2^2}{2m}$, $E_1 = \dfrac{p_1^2}{2m}$, etc. $\Rightarrow E_k = \dfrac{k^2}{2\mu} = \dfrac{k'^2}{2\mu}$ $(\hbar = 1)$ **

- energy-momentum relation for free particle, why not always true? ("virtuality")

- Elastic scattering $E_{in} = E_{out}$. Effective one-body problem.

Show F. Nunes scattering cartoon



all you measure is how many counts in that angular bin

proportional linearly to $d\Omega$, time, incident flux $\Rightarrow$ divide out (normalize)

$$\psi_E^{(+)}(\vec{r}) \xrightarrow[r\to\infty]{} \frac{1}{(2\pi)^{3/2}}\Big[\underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \underbrace{f(k,\theta,\phi)\frac{e^{ikr}}{r}}_{\text{scattered}}\Big]$$

$$\frac{d\sigma}{d\Omega}(k,\theta,\phi) = \frac{\#\text{ scattering into } d\Omega \text{ per time}}{\#\text{ incident per area per time}} = \frac{k/\mu\cdot|f|^2/r^2\cdot r^2}{k/\mu}$$ } from probability current

physics! $\Rightarrow \frac{d\sigma}{d\Omega} = |f(k,\theta,\varphi)|^2 \to |f(k,\theta)|^2$ (no φ dependence here from spin polarization)

Expand: $\psi(r,\theta) = \sum_{l=0}^{\infty} c_l \frac{u_l(r)}{r} P_l(\cos\theta)$ $\leftarrow$ [more generally would use $Y_{lm}(\theta,\varphi)$]

$$\Rightarrow -\frac{1}{2\mu}\frac{d^2u_l}{dr^2} + V(r)u_l + \frac{l(l+1)}{2\mu r^2}u_l = \frac{k^2}{2\mu}u_l \Rightarrow \frac{d^2u_l}{dr^2} - \left(\frac{l(l+1)}{r^2} + V(r) - k^2\right)u_l(r) = 0$$

solve to find scattering

Pick out θ dependence of f: $f(k,\theta) = \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)$ [defines $f_l(k)$]
(central V here)

incoming $e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta) \times (-1)^{l+1}\frac{e^{-ikr} + e^{ikr}}{2ikr}$

(incoming spherical: e^{-ikr}; outgoing spherical: e^{ikr})

outgoing scattering $f_l(k)\frac{e^{ikr}}{r}\cdot\frac{2ik}{2ik}$

$$\boxed{\psi_E^{(+)}(\vec{r}) \xrightarrow[r\to\infty]{} \frac{1}{(2\pi)^{3/2}}\sum_l(2l+1)P_l(\cos\theta)\frac{(-1)^{l+1}e^{-ikr} + S_l(k)e^{ikr}}{2ikr}}$$

$$\Rightarrow \underbrace{[1 + 2ikf_l(k)]}_{S_l(k)}\frac{e^{ikr}}{2ikr}$$

What is physical interpretation of the "1"?

$S_l(k)$ — partial wave S-matrix (warning: different normalizations)

Probability conservation: $|S_l(k)|^2 = 1 \Rightarrow S_l(k) = e^{2i\delta_l(k)} = \frac{e^{i\delta_l(k)}}{e^{-i\delta_l(k)}}$ (not a j idea!)
(elastic) pure phase

- defines phase shift up to multiple of π

3-3

Derive in exercises $f_l(k) = \frac{S_l(k)-1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k} = \frac{1}{k\cot\delta_l - ik}$

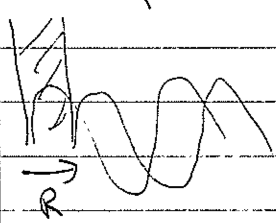
units? $\hbar=1$, $\frac{1}{k}$ is length $\Rightarrow d\sigma \propto |f_l^2| \sim [L]^2$ ✓ we'll see again!

Combined: $\psi_E^{(+)}(\vec r) \underset{r\to\infty}{\longrightarrow} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta)\, i^l e^{i\delta_l} \frac{\sin(kr - l\frac{\pi}{2} + \delta_l)}{kr}$

and $u_l \propto \sin(kr - l\frac{\pi}{2} + \delta_l(k))$

2 min.: If I'm being careful, is the phase shift a function of energy or momentum? [on-shell $\Rightarrow$ either!]

- Ambiguity $\delta_l \to \delta_l + \pi$ or $\delta_l + 2\pi$ or ... $\Rightarrow$ physics is unchanged.
  - Levinson's Theorem $\delta_l(k=0) = (\#\text{ bound states}) * \pi$ } explore numerically in exercise.
    If $\delta_l(k)$ is continuous and $\delta_l(k\to\infty) = 0$
- Show pictures of phase shifts: repulsive pushed out, attractive pulled in



$\leftarrow$ hard sphere $u_0 \propto \sin(kr - kR) \Rightarrow \boxed{\delta_0(k) = -kR}$
S-wave

Think about numerical solution.

$u_0(r) \underset{r\to\infty}{\longrightarrow} \sin(kr + \delta_0(k)) = \cos\delta_0 \sin kr + \sin\delta_0 \cos kr$

$\left[u_l(r) \to \cos\delta_l\, \hat{j}_l(kr) - \sin\delta_l\, \hat{n}_l(kr),\quad \hat{j}_l(z) \equiv \frac{j_l(z)}{z},\ \hat{n}_l(z) \equiv \frac{n_l(z)}{z}\right]$

integrate S-eqn to large enough r (how large?) for given k

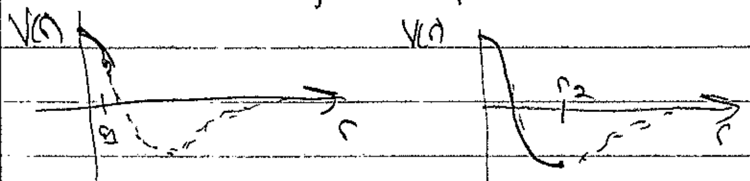
Use $\frac{u_0(r_1)}{u_0(r_2)} \to$ solve for $\tan\delta_l(k)$

or $\frac{u_0'(r_1)}{u_0(r_1)}$ [easy for square well, take $r_1 = R$]

Alternative: Variable Phase Approach (VPA)

given k: • imagine integrating out from the origin: pulled in or out according to V at that point ⇒ accumulate phase shift

• define $\delta(k,r)$ as phase shift at momentum k when potential cut at r



Class: $\delta(k,0) \overset{?}{=} 0$

$\delta(k,\infty) \overset{?}{=} \delta(k)$ (!)

Satisfies diff. eq: $\frac{d}{dr}\delta(k,r) = -\frac{1}{k}[2\mu V(r)]\sin^2[kr+\delta(k,r)]$

- nonlinear 1st order
- look at Mathematica snippet ⇒ easy code! Play with notebook.
- $\sin^2[\,] \geq 0$ always ⇒ what does this say about phase when a potential is attractive or repulsive?
- derivation in notes [fill in details and generalize!]

⇒ Exercises

Non-uniqueness

- inverse scattering idea ⇒ given $\delta_\ell(k)$ for all k, find V(r) (or given $\delta_\ell(k)$ for all ℓ at some k)
  - works if central and no bound states or bound state info given
  - but unitary transformation → infinite "phase equivalent" potentials → same physics. Usually non-local!
- so idea that there is one true potential is misguided

- Unitary transformations
  - $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$ time evolution (also non-unitary $t \to -i\tau$)
  - $U = e^{-i\vec{\alpha}\cdot\vec{G}}$ symmetry transformations
  - unitary transformations of Hamiltonians (e.g. by RG methods)

"Recall" from linear algebra:

$$U^\dagger U = UU^\dagger = I \Rightarrow E_n = \langle \psi_n | \underbrace{\hat{H}}_{=\hat{H}_0+\hat{V}} | \psi_n \rangle = \langle \psi_n | \underbrace{U \hat{H} U^\dagger}_{\tilde{H}} | \tilde{\psi}_n \rangle$$

$U^\dagger U$ $\quad$ $U^\dagger U$ $\quad$ $U|\psi_n\rangle$

If U is short-ranged, then H and $\tilde{H}$ produce same phase shifts, energies

How do you transform $\hat{O}$? To preserve matrix elements $\hat{O} \to \tilde{\hat{O}} = U \hat{O} U^\dagger$.

What quantities are changed? [Exercise question]

- Local and non-local potentials

$\mathbb{1} = \int d^3r\, |\vec{r}\rangle\langle\vec{r}|$ and $\langle \vec{r} | \vec{r}' \rangle = \delta^3(\vec{r} - \vec{r}')$

- Consider $\hat{H} = \frac{\hat{k}^2}{2\mu} + \hat{V}$ and $\langle \psi | \hat{H} | \psi \rangle$

Coordinate space: $\langle \psi | H | \psi \rangle = \int d^3r'\, d^3r\, \langle \psi | \vec{r}' \rangle \langle \vec{r}' | H | \vec{r} \rangle \langle \vec{r} | \psi \rangle$

$$\langle \vec{r}' | \frac{\hat{k}^2}{2\mu} | \vec{r} \rangle = \delta^3(\vec{r}' - \vec{r}) \frac{-\hbar^2 \nabla^2}{2\mu} \qquad \langle \vec{r}' | \hat{V} | \vec{r} \rangle = \begin{cases} V(\vec{r})\, \delta^3(\vec{r}' - \vec{r}) & \text{if local} \\ V(\vec{r}', \vec{r}) & \text{otherwise} \end{cases}$$

See 1-D version on slides

not diagonal

S-eqn. $-\frac{\hbar^2}{2\mu} \nabla^2 \psi(\vec{r}) + V(\vec{r}) \psi(\vec{r}) = E \psi(\vec{r}) \Rightarrow -\frac{\hbar^2}{2\mu} \nabla^2 \psi(\vec{r}) + \int d^3r'\, V(\vec{r}, \vec{r}') \psi(\vec{r}') = E \psi(\vec{r})$

Momentum space: $\langle \psi | H | \psi \rangle = \int d^3k'\, d^3k\, \langle \psi | \vec{k}' \rangle \langle \vec{k}' | H | \vec{k} \rangle \langle \vec{k} | \psi \rangle$ $\quad$ $\mathbb{1} = \int d^3k\, |\vec{k}\rangle\langle\vec{k}|$, $\langle \vec{k} | \vec{k}' \rangle = \delta^3(\vec{k} - \vec{k}')$

$$\Rightarrow \langle \vec{k}' | \frac{\hat{k}^2}{2\mu} | \vec{k} \rangle = \delta^3(\vec{k} - \vec{k}') \frac{\hbar^2 k^2}{2\mu} \qquad \langle \vec{k}' | \hat{V} | \vec{k} \rangle = \begin{cases} V(\vec{k}' - \vec{k}) & \text{if local} \\ V(\vec{k}', \vec{k}) & \text{otherwise} \end{cases}$$

$$\text{Yukawa} \Rightarrow \frac{e^{-m|\vec{r}|}}{4\pi |\vec{r}|} \overset{FT}{\Longleftrightarrow} \frac{1}{(\vec{k} - \vec{k}')^2 + m^2}$$

momentum transfer (not relative momentum)

Now partial wave expansion: (Taylor conventions)

$$\langle \vec{k}'|V|\vec{k}\rangle = \frac{2}{\pi}\sum_{l,m} V_l(k',k)\, Y^*_{lm}(\Omega_{k'})\, Y_{lm}(\Omega_k)$$

(not the same as local)

assuming central potential $\langle k' l' m'|V|k l m\rangle = \delta_{ll'}\delta_{mm'} V_l(k',k)$

- next section – mix different l's with tensor.

hard to tell if local or nonlocal!

S-eqn ⇒ Lippmann-Schwinger equation for T-matrix

$$T^{(+)}(\vec{k}',\vec{k};E) = V(\vec{k}',\vec{k}) + \int d^3q\, \frac{V(\vec{k}',\vec{q})\,T(\vec{q},\vec{k};E)}{E - \frac{q^2}{m} + i\varepsilon}$$ (derive in exercises)

- Expand $\langle \vec{k}'|T^{(+)}(E)|\vec{k}\rangle$ like $\langle \vec{k}'|V|\vec{k}\rangle$

derive in exercises ⇒ $$T_l(k',k;E) = V_l(k',k) + \frac{2}{\pi}\int_0^\infty dq\, q^2\, \frac{V_l(k',q)\,T_l(q,k;E)}{E - E_q + i\varepsilon} \qquad \boxed{E_q = \frac{q^2}{m}}$$

- any k', k, E works here
- but only on-shell related to scattering amplitude $f_l(k)$!

$$T_l(k,k;E=E_k) = -\frac{2\pi}{\mu} f_l(k)$$

- but if we put $k'=k$, $E=E_k$ on left, still need $T_l(q,k;E_k)$ for all $q \neq k$ on right ⇒ half-on shell.

- Operator form: $\hat{T}(z) = \hat{V} + \hat{V}\frac{1}{z-H_0}\hat{T}(z)$

Born series

$$= \hat{V} + \hat{V}\frac{1}{z-H_0}\hat{V} + \hat{V}\frac{1}{z-H_0}\hat{V}\frac{1}{z-H}\hat{V} + \ldots$$

- Take $\langle \vec{k}'|\ \ |\vec{k}\rangle$ matrix element and insert $1 = \int d^3q\, |\vec{q}\rangle\langle\vec{q}|$ to recover full LS equation or $1 = \frac{2}{\pi}\int q^2 dq\, |q\rangle\langle q|$ to get partial wave.

- In exercises: numerical evaluation as matrix equation.

Effective range expansion: 1st pass (we'll see it again!)

Schwinger: $k^{2l+1}\cot\delta_l(k)$ can be expanded in Taylor series in k^2
$\Rightarrow$ effective range expansion or ERE.

· The coefficients have names:

$$l=0 \quad k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{1}{2}r_0k^2 - P r_0^3 k^4 + \ldots$$

a_0 or a_s or a "scattering length"; r_0 or r_s or r_e "effective range"; P "shape parameter"

· Try it for hard-sphere scattering (radius R) $\Rightarrow \delta_0(k) = -kR$

$$k\cot(-kR) \overset{\text{Taylor expansion}}{=} -\frac{1}{R} + \frac{1}{3}Rk^2 + \ldots \Rightarrow a_0 = R,\ r_0 = \frac{2R}{3}$$

note sign

· More general:
  - $r_0 \sim R$, "range" of potential (for Yukawa?)
  - a_0 can be anything
    - if $a_0 \sim R$ then "natural" (c.f. naive dimensional analysis)
    - if $|a_0| \gg R$ (unnatural), then interesting (e.g. neutrons, cold atoms)

· Associate sign and size of a_0 with behavior of scattering wave function as energy (or k) $\to 0$

$$\frac{\sin(kr+\delta_0(k))}{k} \xrightarrow[k\to 0]{} r - a_0 \quad \text{[show this]}$$

· See pictures: a_0 ranges from $-\infty$ to $+\infty$. Large near bound state at zero energy (or just miss)

· low-energy $l=0$, $f_0(k) = -\frac{1}{k\cot\delta_0 - ik} \doteq -\frac{1}{-1/a_0 - ik} \Rightarrow \sigma(k) = \frac{4\pi}{1/a_0^2 + k^2}$ ($4\pi|f_0(k)|^2$)

natural: $|ka_0| \ll 1$: $\frac{d\sigma}{d\Omega} = a_0^2 \Rightarrow \sigma = 4\pi a_0^2$ unnatural: $|ka_0| \gg 1$: $\frac{d\sigma}{d\Omega} \to \frac{1}{k^2} \Rightarrow \sigma = \frac{4\pi}{k^2}$ "unitary limit"