4. Nuclear Forces 1

Recaps and follow-ups from 1.3. Scattering Theory 1.

- Comments on working together
  - teaching each other is a win-win situation.
    cf. Weinberg's insights from teaching in history of EFT essay.
  - be sure to talk to more than the same few people and take advantage of Piazza.
  - if anyone is feeling left out, please let us know!

- Follow-ups from Scattering 1 exercise: "Exploring the Lippmann-Schwinger" equation:

$$|\psi_E^+\rangle = |\vec{k}\rangle + \frac{1}{E-H_0+i\varepsilon} V |\psi_E^+\rangle$$

is just a rewrite of the Schrödinger equation as an integral equation

$\frac{1}{E-H_0+i\varepsilon}$ is just the Green's function.

Check by operating with $E-H_0$ (to kill denominator)

$$\Rightarrow (E-H_0)|\psi_E^+\rangle = \overset{0}{\cancel{(E-H_0)|\vec{k}\rangle}} + V|\psi_E^+\rangle \quad \checkmark$$

since $(H_0+V)|\psi_E^+\rangle = E|\psi_E^+\rangle$

But now $\underbrace{V|\psi_E^+\rangle}_{T^{(+)}(E_k)|\vec{k}\rangle} = V|\vec{k}\rangle + V\frac{1}{E-H_0+i\varepsilon}\underbrace{V|\psi_E^+\rangle}_{T^{(+)}(E_k)|\vec{k}\rangle}$ only on-shell

Hit with $\langle\vec{k}'|$ to get LS equation we discussed.

Or, generalize to an operator $\hat{T}(z)$ with z any complex number

$$\hat{T}(z) = \hat{V} + \hat{V}\frac{1}{z-\hat{H}_0}\hat{T}(z) = \hat{V} + \hat{V}\frac{1}{z-\hat{H}_0}\hat{V} + \hat{V}\frac{1}{z-\hat{H}_0}\hat{V}\frac{1}{z-\hat{H}_0}\hat{V} + \dots$$

where the 2nd equality follows by iteration.

Take $z = E_k + i\varepsilon$, sandwich $\langle\vec{k}'|, |\vec{k}\rangle$, we get back the full LS. $\Rightarrow$ generalized.

Separable potential $\hat{V} = g|\eta\rangle\langle\eta|$

to be local it would have to be function of $|\vec{k}-\vec{k}'|$. It's not!

$\Rightarrow \langle\vec{k}|\hat{V}|\vec{k}'\rangle = g\langle\vec{k}|\eta\rangle\langle\eta|\vec{k}'\rangle$

$\langle\vec{r}|\hat{V}|\vec{r}'\rangle = g\langle\vec{r}|\eta\rangle\langle\eta|\vec{r}'\rangle$

not local! (unless $\eta(\vec{r}) \propto \delta^3(\vec{r})$)

would have to be $\delta^3(\vec{r}-\vec{r}')$ for local

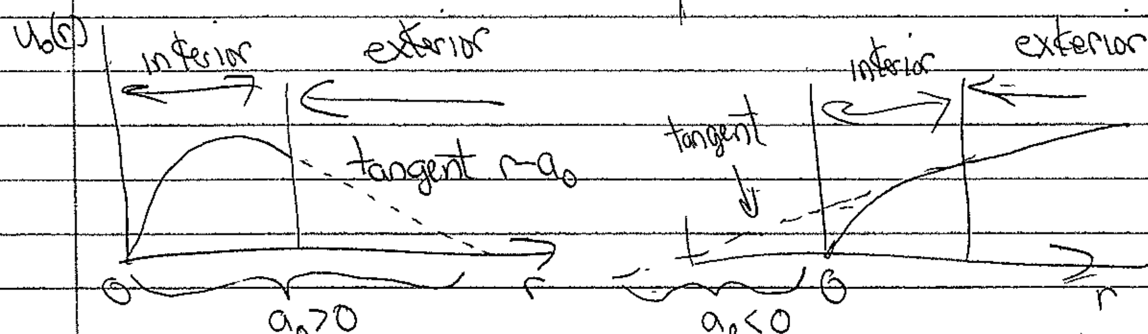
$\Rightarrow$ can solve for $\hat{T}(z)$ algebraically! $\Rightarrow$ exercise.

- Effective range expansion

. recall $f_\ell(k) = \dfrac{S_\ell(k)-1}{2ik} = \dfrac{1}{k\cot\delta_\ell(k) - ik}$

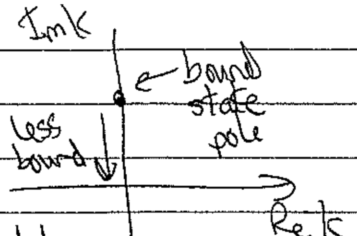
. for $\ell=0$, $k\cot\delta_0(k)$ has Taylor series (radius of convergence?)

⋆ $\Rightarrow$ go to 3-7 and do that page.



[figure 5 in 3-1 notes has opposite sign convention for a_0!]

- Note that $f_0(k) = -\dfrac{1}{-1/a_0 - ik}$ has a pole at $k = \frac{i}{a_0}$ (imaginary axis)



$\Rightarrow E \approx -\dfrac{\hbar^2|k|^2}{2\mu} = -\dfrac{\hbar^2}{2\mu a_0^2}$

pure imaginary momentum so

recommended:

[Problem Scattery-1 exercises shows square well pole $\Rightarrow$ same eigenvalue condition]

- Taylor expansion for ERE! radius of convergence?

- Show figures of nuclear force potential

- Comments:
  - eight decade effort to accurately describe nuclear force, starting from Yukawa's meson theory in 1935
  - pion was always a part, but chiral symmetry not understood until much later (Gerry Brown was key)
  - but how should we relate to QCD?
    - isn't left picture more accurate ⇒ better???
  - brief history (as in longer notes)
  - Comments:
    - model independence, theory error estimates ⇒ goal
    - quark substructure is included even though point nucleons (cf. multipole expansion)
    - future: find low-energy constants (LEC's) from lattice QCD! (next week)
    - if we just want to do low-energy physics, explicit quarks and gluons can be gross overkill.

- Look next at general constraints on NN interactions
  - notes are more extensive but this is a standard exercise
    ⇒ we're interested in the ideas more than the details here
  - we talked a bit about non-uniqueness and non-local potentials ⇒ let's follow up.

· One way to proceed is to start with a general basis in spin, isospin, spatial coordinates and identify symmetry constraints.

· So give all matrix elements $\langle \vec{r}_1' s_1' t_1' \vec{r}_2' s_2' t_2' | \hat{V} | \vec{r}_1 s_1 t_1 \vec{r}_2 s_2 t_2 \rangle$ with $s_i = \pm 1/2$, $t_i = \pm 1/2$ are spin and isospin projections
· complete basis.

· Suppress spin, isospin for a moment

$$\hat{V} | \vec{r}_1 \vec{r}_2 \rangle = \int V(\vec{r}_1', \vec{r}_2', \vec{r}_1, \vec{r}_2) | r_1' r_2' \rangle d^3 r_1' \, d^3 r_2'$$

local means $V = V(\vec{r}_1, \vec{r}_2) \delta^3(\vec{r}_1 - \vec{r}_1') \delta^3(\vec{r}_2 - \vec{r}_2') \Rightarrow \hat{V} | \vec{r}_1 \vec{r}_2 \rangle = V(\vec{r}_1, \vec{r}_2) | \vec{r}_1 \vec{r}_2 \rangle$
· only depends on positions, not velocities of particles.

$\Rightarrow$ suggests that non-locality is related to velocity (or momentum) dependence.

· Basic result from $|\vec{r}_1' \vec{r}_2' \rangle = |\vec{r}_1 \vec{r}_2 \rangle + [(\vec{r}_1' - \vec{r}_1) \cdot \vec{\nabla}_1 + (\vec{r}_2' - \vec{r}_2) \cdot \vec{\nabla}_2] |\vec{r}_1 \vec{r}_2 \rangle + \ldots$ ($\propto \vec{p}_1, \vec{p}_2$)

$$\Rightarrow \hat{V} | \vec{r}_1 \vec{r}_2 \rangle = \int V(\vec{r}_1', \vec{r}_2', \vec{r}_1, \vec{r}_2) \left(e^{i(\vec{r}_1' - \vec{r}_1) \cdot \vec{p}_1} + e^{i(\vec{r}_2' - \vec{r}_2) \cdot \vec{p}_2} \right) | \vec{r}_1 \vec{r}_2 \rangle \, d^3 r_1' \, d^3 r_2'$$

do the $\vec{r}_1', \vec{r}_2'$ integrals $= \hat{V}(\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2) | \vec{r}_1 \vec{r}_2 \rangle$ so have traded for momentum (operator) dependence

$\Rightarrow$ expand to low order in $\vec{p}_1, \vec{p}_2$ (zeroth order is local)

· Much too general a form $\Rightarrow$ symmetry constraints.
· In general, we associate a Hermitian generator $\vec{G}$ with a symmetry and unitary

$$U = e^{-i\vec{\alpha} \cdot \vec{G}}$$

$\leftarrow \vec{\alpha} \cdot \vec{G}$ is shorthand for any scalar — could be tensors

with $|\psi_i' \rangle = U |\psi_i \rangle$ with the transformation

Given an operator (like the Hamiltonian), symmetry means

$$\langle \psi_i' | \hat{O} | \psi_j' \rangle = \langle \psi_i | U^\dagger \hat{O} U | \psi_j \rangle = \langle \psi_i | \hat{O} | \psi_j \rangle$$

$$\Rightarrow U^\dagger \hat{O} U = \hat{O} \Rightarrow [\hat{O}, U] = 0.$$

Claim (shown in exercises): we only need to look at $[\hat{O}, \hat{\vec{G}}_i] = 0$

Example: we claim that if isospin is a good symmetry, then potential takes form

$$V = \alpha_I + \beta_I \vec{\tau}_1 \cdot \vec{\tau}_2 \qquad \text{"class I" in Henley-Miller scheme}$$

all other spin-space dependence

Here $\vec{G} \to \vec{T} = \frac{1}{2}(\vec{\tau}_1 + \vec{\tau}_2)$, the total isospin (cf. angular momentum is the generator of rotations).

$$[\vec{\tau}_1 \cdot \vec{\tau}_2, \vec{T}]_j = \frac{1}{2}[\tau_{1i}\tau_{2i}, \tau_{1j} + \tau_{2j}] = \frac{1}{2}\underbrace{[\tau_{1i}, \tau_{1j}]}_{2i\varepsilon_{ijk}\tau_{1k}}\tau_{2i} + \frac{1}{2}\tau_{1i}\underbrace{[\tau_{2i}, \tau_{2j}]}_{2i\varepsilon_{ijk}\tau_{2k}}$$

$= 0$ by antisymmetry of ε_{ijk}

two minutes: is $\vec{\tau}_1 \cdot \vec{\tau}_2 = \vec{\tau}_2 \cdot \vec{\tau}_1$? (exercise)

- One can consider other combinations of $\vec{\tau}_1$ and $\vec{\tau}_2$, but none works for full isospin dependence. How about $(\vec{\tau}_1 \cdot \vec{\tau}_2)^2$? (Exercise)
  - Later we'll come back to having not full isospin symmetry, but only charge symmetry [$a_{pp} \doteq a_{nn} \neq a_{np}$ when Coulomb removed]

$$P_{CS} = e^{i\pi T_2} \quad \text{with } \vec{T} = \frac{1}{2}(\vec{\tau}_1 + \vec{\tau}_2) \text{ again} \Rightarrow \text{only rotate about } y \text{ axis in isospin}$$

$$\Rightarrow P_{cs}|u\rangle = -|d\rangle, \quad P_{cs}|d\rangle = |u\rangle$$

Then $V_{II} = \alpha_{II} \tau_{1z}\tau_{2z}$ works (recall $\tau_2 = \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$)

Called charge independence breaking or CIB.

[so $pp \doteq nn \neq np$ with V_{II}]

Henley and Miller considered all possible classifications (more later)

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Apply symmetry constraints

1. $\hat{V}$ hermitian
2. $V(1,2) = V(2,1)$ identical particles
3. translational invariance $U = e^{-i\alpha \cdot \vec{P}}$ $\Rightarrow [\vec{P}, \hat{V}] = 0$

   $\Rightarrow$ not separately $\vec{r}_1, \vec{r}_2$ but $\vec{r}_1 - \vec{r}_2 \rightarrow \vec{r}$
4. Galilean invariance

   $e^{-iM\vec{u}\cdot\vec{R}}$ $\quad \vec{r}_i' = \vec{r}_i$ $\quad M = \sum_i m_i \quad \vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i$

   $\quad p_i' = p_i - m_i \vec{u}$

   $\Rightarrow \vec{p}_1, \vec{p}_2 \rightarrow \vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ only
5. rotational invariance : $\vec{r}^2, \vec{p}^2, \vec{r}\vec{p} + \vec{p}\vec{r} \rightarrow \vec{L}^2, \vec{L}\cdot\vec{S}$
6. parity $\vec{r}_i' = -\vec{r}_i$, $\vec{k}_i' = -\vec{k}$, $\sigma_i' = \sigma_i$, $\tau_i' = \tau_i$
7. time-reversal $\vec{r}_i' = \vec{r}_i$, $\vec{k}_i' = -\vec{k}$, $\sigma_i' = -\sigma_i$, $\tau_i' = \tau_i$
8. Baryon and number conservation
9. Isospin charge symmetry

In end $V_{NN} = V_1(\vec{r}, \vec{p}, \vec{\sigma}_1, \vec{\sigma}_2) + V_\tau(\vec{r}, \vec{p}, \vec{\sigma}_1, \vec{\sigma}_2)\vec{\tau}_1 \cdot \vec{\tau}_2$
with V_1 and V_τ classified as

- central: $V_1(\vec{r}, \vec{p}) + V_\sigma(\vec{r}, \vec{p})\vec{\sigma}_1 \cdot \vec{\sigma}_2$
- vector: $V_{LS}(\vec{r}, \vec{p})\vec{L}\cdot\vec{S}$
- tensor: $V_T(\vec{r}, \vec{p}) S_{12}(\hat{r})$ $\quad S_{12}(\hat{r}) \equiv (\hat{r}\cdot\vec{\sigma}_1 \, \hat{r}\cdot\vec{\sigma}_2 - \frac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2)$

In coordinate space

$$\{1_{spin}, \vec{\sigma}_1\cdot\vec{\sigma}_2, S_{12}(\hat{r}), S_{12}(\hat{p}), \vec{L}\cdot\vec{S}, (\vec{L}\cdot\vec{S})^2\} \times \{1_{isospin}, \vec{\tau}_1\cdot\vec{\tau}_2\}$$

$\times$ scalar functions of $\vec{r}^2, \vec{p}^2, \vec{L}^2$

In momentum space

$$\{1_{spin}, \vec{\sigma}_1\cdot\vec{\sigma}_2, S_{12}(\hat{q}), S_{12}(\vec{k}), i\vec{S}\cdot(\vec{q}\times\vec{k}), \sigma_1\cdot(\vec{q}\times\vec{k})\, \sigma_2\cdot\vec{q}\times\vec{k}\} \times \{\ \}$$

where $\vec{q} \equiv \vec{p}' - \vec{p}$, $\vec{k} = (\vec{p}' + \vec{p})/2$ times scalar functions of $\vec{p}^2, \vec{p}'^2, \vec{p}'\cdot\vec{p}$

4.7

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NN
Laundry list of $_\wedge$ interactions available

i) high precision phenomenological models : $\chi^2/\mathrm{dof} \approx 1$
- boson exchange
- AV18
- inverse scattering

ii) chiral EFT $\rightarrow$ later this week and next week

iii) "Toy" NN potentials
- Minnesota model : central sum of 3 Gaussians
- Malfliet-Tjon potential : central sum of Yukawas

Boson exchange : start with covariant form

$$V(1,2) = \left(\frac{-g_s^2}{4\pi}\right)\frac{e^{-m_s r_{12}}}{r_{12}} + \gamma_1^\mu \gamma_{2\mu}\left(\frac{g_w^2}{4\pi}\right)\frac{e^{-m_w r_{12}}}{r_{12}} + \gamma_1^5\gamma_2^5\,\vec{\tau}_1\cdot\vec{\tau}_2\left(\frac{g_\pi^2}{4\pi}\right)\frac{e^{-m_\pi r_{12}}}{r_{12}} + \ldots$$

evaluated between 4 component spinors

$$U(\vec{p}) \propto \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\chi \end{pmatrix} \qquad \chi = \text{Pauli spinor}$$

$\Rightarrow$ reduction to two component form has rich structure,
- model dependent but successful phenomenologically

AV18 : 18 refers to 18 operators

$$\{1, \vec{\sigma}_1\cdot\vec{\sigma}_2, S_{12}, \vec{L}\cdot\vec{S}, \vec{L}^2, L^2\vec{\sigma}_1\cdot\vec{\sigma}_2, (\vec{L}\cdot\vec{S})^2\} \otimes \{1, \vec{\tau}_1\cdot\vec{\tau}_2\} \quad 14$$

- + CD and CSB terms (4 of them)

- $V_{EM} + V_\pi + V_{\text{short-range}}$

- V_π is one pion exchange $\propto f^2(\vec{\tau}_1\cdot\vec{\tau}_2)\left[\left(3\,\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right)\left(1+\frac{3}{m_\pi r}+\frac{3}{(m_\pi r)^2}\right) + \vec{\sigma}_1\cdot\vec{\sigma}_2\right] \times \frac{e^{-m_\pi r}}{r}$

- V_π multiplied by $1-e^{-cr^2}$ and $(1-e^{-cr^2})^2$ for tensor part
- short range $\times$ $W(r) = [1+e^{(r-r_0)/a}]^{-1}$

Problems with phenomenological potentials

- usually have very strong repulsive short-range part that is a problem for (some) types of) many-body calculations
- difficult to estimate theoretical error in a calculation and the range of applicability (where should in fail?)
- Three-nucleon forces are not systematically included. How to define consistent 3NF's and operators?
- * largely unconnected to QCD (eg. only partial chiral symmetry)
  - don't connect NN and πN
  - can't connect to lattice QCD

$\Rightarrow$ effective (field) theory

* - Look at quotes from Georgi $\rightarrow$ examples that take a parameter to zero or ∞ in long notes.

Principles of low-energy effective theories

- If a system is probed at low-energies, fine details are not resolved
- Use low-energy variables for low-energy processes (easier, more efficient, ...)
- Replace the unresolved short-distance structure by something simpler (and wrong at short distances) $\Rightarrow$ without distorting low-energy observables. Renormalization!
- EFT does this systematically (i.e., in a expansion)

- cf. multipole expansion