6. Nuclear Forces 2

a. Recap: Principles of low-energy effective theories

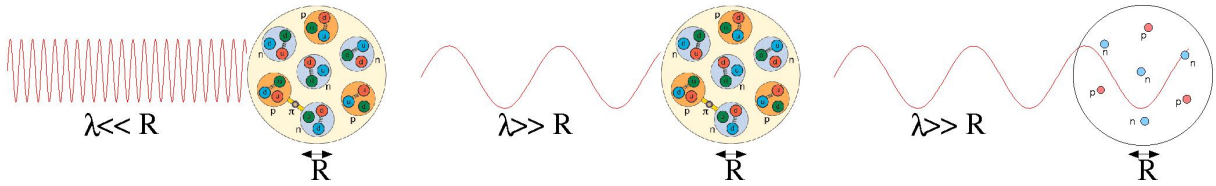


Figure 1: Left: high-resolution, with wavelength of probe short compared to characteristic size of probed structure. Middle: low-resolution probe doesn't resolve the details (because of diffraction). Right: low-energy theory takes advantage and replaces short-distance with long-distance degrees of freedom.

At the end of Section 4, a summary of basic principles for low-energy effective theories was given, which we recap here.

- If system is probed at low energies, fine details are not resolved. (Note that "not resolved" doesn't mean that the details are absent, only that the detailed features are washed out.)
- So we can use low-energy variables ("degrees of freedom") for low-energy processes (as they can be easier, more efficient, etc.).
- The short-distance structure can be *replaced* by something simpler in this way (even though it is wrong at short distances!) without distorting low-energy observables. It is important that being wrong at short distances ("incorrect UV behavior") doesn't matter for the long distance observables (after renormalization), but also cautions us that success for long distances doesn't argue in favor of the correctness of our description of short distance observables.
- This plan is systematically achieved by effective field theory.

In this section and next we want to apply these ideas to the simplest possible example, for which the resolution is so low we don't resolve *any* interaction between the particles. For the nuclear case, this means that the exchange of pions is considered short-ranged; that is the pion is a heavy degree of freedom that is replaced (as are all other interactions) by contact interactions.

In this section we'll lay out the basic ideas of renormalization and power counting for the simplest case: a natural scattering length. In the next section, the more interesting case for nuclear physics of an unnatural scattering length will be explored in detail.

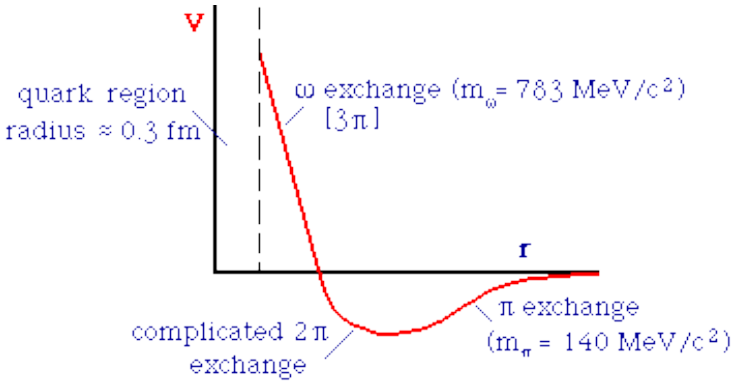


Figure 2: Ranges associated with boson exchange. The intermediate two-pion exchange attraction can be simulated with isoscalar, scalar exchange ("σ").

b. Pionless effective field theory

b.1 Low-energy simplification of boson exchange interactions

Let's first get some intuition about what NN forces would look like at low-energies. For this purpose, it is instructive to start with boson exchange potentials. Consider Fourier transforming to momentum space a Yukawa potential with mass μ arising from one of the boson exchanges considered in the Nuclear Forces 1 section (without paying attention to the overall normalization),

$$g^2 \frac{e^{-\mu|\mathbf{r}|}}{4\pi|\mathbf{r}|} \longleftrightarrow \frac{g^2}{(\mathbf{k}'-\mathbf{k})^2+\mu^2} \,, \tag{1}$$

which manifestly depends on $\mathbf{k}^2$, $\mathbf{k}'^2$, and $\boldsymbol{k}\cdot\boldsymbol{k}'$. (Note: we are ignoring any spin or isospin structure here.) Now suppose we are at very low resolution, which means low momenta $k, k' \ll \mu$ (the deBroglie wavelengths are long). We can Taylor expand in Eq. (1):

$$A_0 + A_2(\mathbf{k}^2+\mathbf{k}'^2) + A_2'\mathbf{k}\cdot\mathbf{k}' + \cdots \tag{2}$$

These terms are called *contact interactions* because when Fourier transformed to coordinate space they become delta functions in the interparticle distance and derivatives of such delta functions. So this implies that at low resolution we can replace an extended interaction (in this case, from the exchange of a particle) with a series of contact interactions.

If all we ever did was first-order perturbation theory, then a simple Taylor expansion like this is all we need and we could pick the A_i's to match the corresponding Taylor expansion of the Yukawa term-by-term to reproduce the results of the Yukawa theory. But what if we do second-order perturbation theory (or include the effects of all orders by solving the Schrödinger equation)? Then the sums over intermediate states become integrals over the momenta, which then contain parts where for which $k, k' \geq \mu$ and the expansion does not apply (i.e., it is wrong at high energies). What do we do? Stay tuned!

b.2 Recap of effective range expansion

Because we are considering the very low-energy limit of a theory (i.e., for momenta below any exchanged particle mass), we expect we should reproduce the physics of the effective range expansion discussed in earlier sections. This is generically called the "pionless" EFT, because there are *no* long-range degrees of freedom (i.e., the pion is treated as heavy with $k \ll m_\pi$ rather than being treated as very light because of its Goldstone boson nature). Recall that the partial wave expansion of the scattering amplitude or the on-shell T-matrix (which is only different by a multiplicative factor) takes the form

$$f(k,\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta) = \sum_{k=0}^{\infty} \frac{2l+1}{k\cot\delta_l(k) - ik} P_l(\cos\theta) \ . \tag{3}$$

The combination $k\cot\delta_0(k)$ [and $k^{2l+1}\cot\delta_l(k)$ for general l] can be expanded in a power series in k^2 (the effective range expansion or ERE), which for $l=0$ is written

$$k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 - P r_0^3 k^4 + \cdots \ , \tag{4}$$

which defines the S-wave *scattering length* a_0, the S-wave *effective range* r_0 and the S-wave shape parameter P (often the first two are written a_s and r_s or a and r_e). We observed that for a potential of range R, we could have $a_0 \approx R$, which was called natural, or at the other extreme $|a_0| \gg R$ is the unitarity regime. Here we consider only how to reproduce the ERE in the natural case.

b.3 In search of a perturbative expansion

If a_0 is natural, then low-energy scattering in Eq. (3) simplifies further. Let's take the hard sphere potential as a concrete example. The ERE in this case is indeed natural (recall that $\delta(k) = -kR$):

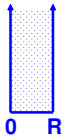
$$k\cot(-kR) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \cdots = -\frac{1}{R}\Big[1 - \frac{(kR)^2}{3} - \frac{(kR)^4}{45} + \cdots\Big] \ , \tag{5}$$

and so if kR is small, we have a convergent expansion. This means that the scattering amplitude itself also has a perturbative expansion in k:

$$\begin{aligned} f_0(k) &= \frac{1}{k\cot\delta(k) - ik} = \frac{1}{-\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \cdots - ik} = \frac{-a_0}{1 - \frac{1}{2} a_0 r_0 k^2 + \cdots + i a_0 k} \\ &\longrightarrow -a_0\big[1 - i a_0 k - (a_0^2 - a_0 r_0/2) k^2 + \mathcal{O}(k^3 a_0^3)\big] \\ &\longrightarrow -R[1 - ikR - 2k^2R^2/3 + \mathcal{O}(k^3R^3)] \ , \end{aligned} \tag{6}$$

where the second line is general (when an ERE is valid) and in the last line we substituted the particular case of hard-sphere scattering. That is, for scattering at momentum $k \ll 1/R$, we should recover a perturbative expansion in kR for the hard-sphere scattering amplitude (because a_0 and r_0 are of order R). Note that the perturbative expansion is *not* an expansion in the strength of the original potential (e.g., the hard sphere).

Can we reproduce this simple expansion for the hard-sphere potential? As just noted, ordinary perturbation theory in the original hard-sphere potential manifestly won't work:



$$\Longrightarrow \quad \langle \mathbf{k}|V|\mathbf{k}'\rangle \propto \int d\mathbf{x}\; e^{i\mathbf{k}\cdot\mathbf{x}}\, V(\mathbf{x})\, e^{-i\mathbf{k}'\cdot\mathbf{x}} \longrightarrow \infty\,, \tag{7}$$

so even first-order perturbation theory fails and higher orders just get worse. The standard solution is to first solve the scattering problem *nonperturbatively*, then expand the amplitude in kR. For our example, this is easy: just use $\delta_0(k) = -kR$ and leave it to Mathematica to find the last line in Eq. (6) by Taylor expansion. Now this is easy to do for 2–2 scattering, but not for the many-body problem! So even for this simple system we would like a more systematic approach. In anticipation of this more interesting application to finite density, we consider the EFT approach: the condition $k \ll 1/R$ means that we probe at low resolution (long wavelengths), so we can replace the potential with a simpler but general interaction.

b.4 EFT for a "natural" short-range interaction

As we motivated when considering boson exchange, a general low-energy expansion in momentum space is:

$$\langle \mathbf{k}|V_{\rm eft}|\mathbf{k}'\rangle = C_0 + \frac{1}{2}C_2(\mathbf{k}^2 + \mathbf{k}'^2) + C_2'\mathbf{k}\cdot\mathbf{k}' + \cdots\,. \tag{8}$$

Recall the conditions we set on a nuclear interaction: hermiticity, invariance under particle 1 $\longleftrightarrow$ particle 2, translational, Galilean boost, and rotational invariance, parity and time-reversal invariance, conservation of baryon number, isospin invariance. These tell us that this expansion really is the most general to this order (in the exercises you can explore the constraints on the k^4 terms). That is, despite our motivation, we are not restricted to the model that the interaction comes from boson exchange, but rather we have a complete expansion that accommodates *any* underlying physics. That is why we call this expansion *model independent*. At this point we are not considering spin, but we'll return briefly to this question below.

In field theory language, we can write the EFT Lagrangian density $\mathcal{L}_{\rm eft}$ with the most general local (contact) interactions (again, not including spin-dependent interactions):

$$\begin{aligned}\mathcal{L}_{\rm eft} = \psi^\dagger\Big[i\frac{\partial}{\partial t} + \frac{\overrightarrow{\nabla}^2}{2m}\Big]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 + \frac{C_2}{16}\big[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{ h.c.}\big] \\ + \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots\end{aligned} \tag{9}$$

where "h.c." stands for hermitian conjugate and $\overleftrightarrow{\nabla}$ is the Galilean invariant derivative:

$$\overleftrightarrow{\nabla} \equiv \overleftarrow{\nabla} - \overrightarrow{\nabla}\,. \tag{10}$$

Using this derivative ensures that the Lagrangian is unchanged if all of the particle momenta are boosted by $\boldsymbol{v}$: $\mathbf{p} \to \mathbf{p} + m\mathbf{v}$.

[When this Lagrangian is used in the literature, it often is accompanied by words like: "this is a general but not unique form of the Lagrangian for short-range spin-independent interactions." This may sound contradictory, but what it means is that there are other choices that can be reached by "field redefinitions," which are basically changes of variable for ψ. These other choices lead to different but physically equivalent forms (e.g., with more time derivatives). Physically equivalent means that you will get the same result in any calculation of measurable quantities. So if you wrote down every possible term allowed by symmetry, it would be over-complete; we can then use field redefinitions to remove the redundancies.]

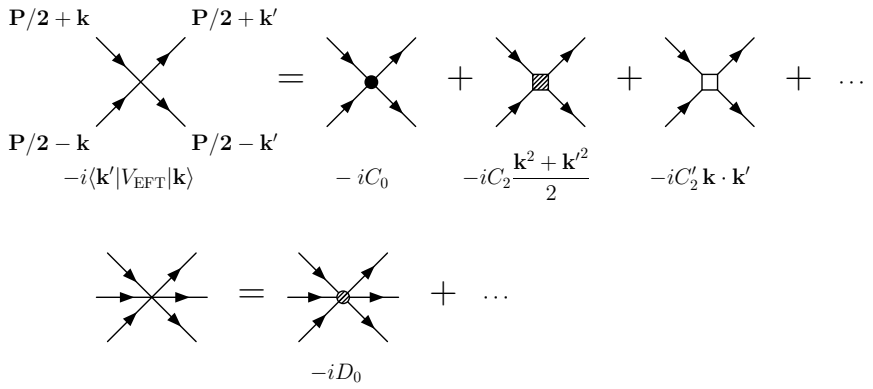


Figure 3: Feynman rules for spinless interaction. The leading two-body and three-body vertices are given.

The Feynman rules for these vertices are given in Fig. 3. Note how the factorial factors introduced in the Lagrangian are canceled in the rules and how we end up with relative momenta (with factors of i) from the derivatives. Those of you with some field theory experience might try deriving these rules, for example from a path integral representation (e.g., the i in front of each comes from the i in e^{iS}). We can understand the factors of relative momentum if we imagine substituting second quantized field expansions for the ψ's and $\psi^\dagger$'s:

$$\psi \longrightarrow \widehat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}\lambda} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \eta_\lambda a_{\mathbf{p}\lambda} \qquad \psi^\dagger \longrightarrow \widehat{\psi}^\dagger(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}\lambda} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \eta_\lambda^\dagger a_{\mathbf{p}\lambda}^\dagger \,, \tag{11}$$

where we are in volume V, λ is the spin projection with two-component spinor η_λ and $a^\dagger$ and a are creation and destruction operators, respectively. Each of the operators is associated with one leg of the scattering; the derivatives ensure that the total momentum $\mathbf{P}$'s are canceled and the hermiticity ensures that the potential is symmetric in $\mathbf{k}$ and $\mathbf{k}'$. There are also combinatoric factors because of the freedom of assigning operators to the legs (this gives factors of 2 to each of the two-body terms and a $3! = 6$ factor to the three-body term).

Let's reproduce the $l = 0$ term in $T(k, \cos\theta)$ (which we'll call T_0) in perturbation theory, which is the same as calculating the Born series for the Lippmann-Schwinger equation,

$$T_0(E) = V_{l=0} + V_{l=0} \frac{1}{E - H_0 + i\epsilon} V_{l=0} + V_{l=0} \frac{1}{E - H_0 + i\epsilon} V_{l=0} \frac{1}{E - H_0 + i\epsilon} V_{l=0} + \ldots \tag{12}$$

to reproduce the effective range expansion:

$$T_0(k) = \frac{4\pi a_0}{m}\left[1 - ia_0 k - (a_0^2 - a_0 r_0/2)k^2 + \mathcal{O}(k^3 a_0^3)\right] \ . \tag{13}$$

Again, at very low momentum (energy), the scattering is described to good accuracy by specifying just the scattering length a_0. The higher-order corrections require r_0 and a_p (for $l = 1$, and so on. We've pulled out a factor of $4\pi a_0/m$ to make the perturbative expansion manifest.

Consider the leading potential $V_{\rm EFT}^{(0)}(\mathbf{x}) = C_0\delta(\mathbf{x})$ or

$$\langle \mathbf{k}|V_{\rm eft}^{(0)}|\mathbf{k}'\rangle \implies \quad \implies C_0 \tag{14}$$

Choosing $C_0^{(0)} = 4\pi a_0/m$ gets the first term. (We've added a superscript (0) in anticipation that this value of C_0 will be modified order-by-order in the expansion.) The next term is $\langle \mathbf{k}|VG_0V|\mathbf{k}'\rangle$ (where $G_0 \equiv 1/(E - H_0 + i\epsilon)$):

$$\implies C_0^{(0)} m \int \frac{d^3q}{(2\pi)^3}\frac{1}{k^2 - q^2 + i\epsilon} C_0^{(0)} \longrightarrow \infty! \tag{15}$$

We can get this result by direct substitution and inserting a complete set of momentum $|\mathbf{q}\rangle$ states (with a different normalization than we've used in earlier sections) or from the field theory perspective one includes a propagator for each internal line and then integrates over the frequency. [Note: you'll work out the details of the latter calculation eventually as a Piazza problem.]

But the result is infinite! This infinite result is called a "linear divergence" because when we put an upper limit on q of Λ_c, then the high-q part of the integral in Eq. (15) is proportional to one power of Λ_c: $\int_c^\Lambda dq\, q^2/q^2 \propto \Lambda_c$. We can extract the leading dependence in a power series in k without actually evaluating the integral:

$$\begin{aligned} I_0(k,\Lambda_c) &\equiv m\int^{\Lambda_c}\frac{d^3q}{(2\pi)^3}\frac{1}{k^2 - q^2 + i\epsilon} = -\frac{m}{2\pi^2}\int_0^{\Lambda_c} dq + \frac{mk^2}{2\pi^2}\int_0^{\Lambda_c}\frac{dq}{k^2 - q^2 + i\epsilon} \\ &= -\frac{m}{2\pi^2}\Lambda_c + \frac{mk^2}{2\pi^2}\mathcal{P}\int_0^{\Lambda_c} dq\,\frac{1}{k^2 - q^2} - i\pi\frac{mk^2}{2\pi^2}\int_0^{\Lambda_c} dq\,\delta(k^2 - q^2) \\ &= -\frac{m}{2\pi^2}\Lambda_c\left(1 + \mathcal{O}\Big(\frac{k^2}{\Lambda_c}\Big)\right) - \frac{ikm}{4\pi} \end{aligned} \tag{16}$$

where we wrote $q^2 = (q^2 - k^2) + k^2$ to get the first equality, used

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \tag{17}$$

to get the second equality, and used

$$\int_0^{\Lambda_c} dq\,\delta(k^2 - q^2) = \frac{1}{2k}\int_0^{\Lambda_c} dq\,\delta(k - q) = \frac{1}{2k}\ , \tag{18}$$

to get the final result. Note that the Λ_c part of the integral comes from the high-momentum part, where $q \gg k$. In contrast, the imaginary part, which is independent of Λ_c, comes from where $q \approx k$.

Now we can absorb the linear Λ_c dependence by adjusting the value of C_0; this is renormalization! In particular, take

$$C_0^{(0)} + C_0^1 = \frac{4\pi a_0}{m} + (C_0^0)^2 \frac{m}{2\pi^2}\Lambda_c = \frac{4\pi a_0}{m}\left(1 + \frac{2a_0\Lambda_c}{\pi} + \cdots\right) \tag{19}$$

and the linear Λ_c dependence is removed as the contribution from $C_0^{(1)}$ in the contact term cancels the leading Λ_c part of the loop integral.

To summarize, when we sum over intermediate states, the momentum of the states get arbitrarily high. The vertices are certainly not correct for that momentum. But we can fix the problem by noting that these vertices *are* correct for low momentum, and for high momentum the intermediate state is at high energy, so it is *highly virtual*. Therefore it can only last a short time and the two vertices are not far apart, so the high-momentum part acts like a local vertex. We can "fix" the incorrect part by just adjusting ("renormalizing") the values of the constants C_0, C_2, and so on.

b.5 Alternative to sharp cutoff: Dimensional regularization with minimal subtraction

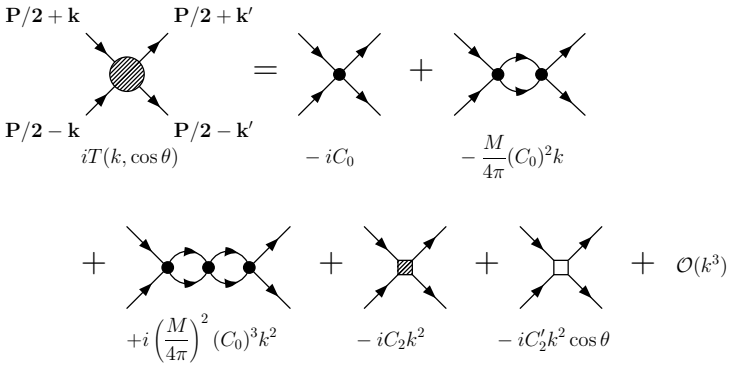
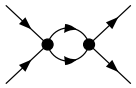


Figure 4: Effective range expansion for a natural scattering length. [Note: This diagram is from a paper in which a different sign convention for T is used than here. In our notation, the left side is $-iT(k, \cos\theta)$.]

Dimensional regularization with minimal subtraction is a cleaner way to regulate and renormalize this EFT than using a cutoff because there will be only one power of k per diagram! We won't be able to use it in more general cases (as when the pion is a long-range degree of freedom) but it is instructive to see how it works here (without filling in too many details). It is based on the integral:



$$\Longrightarrow\ I_0(k) \equiv m\int \frac{d^Dq}{(2\pi)^3}\frac{1}{k^2 - q^2 + i\epsilon} \underset{D\to 3}{\longrightarrow} -\frac{ik}{4\pi}m\,, \tag{20}$$

which we present without derivation. This leads to very simple *power counting*, which in this context means an association with each diagram of a definite power of the expansion parameter following these rules:

- Each propagator contributes M/k^2 because of the energy denominator;
- each d^4k loop: k^5/M (the d^3k integration contributes k^3 while the dk_0 integration picks up a pole of the form $k_0 - k^2/M$, so this counts as k^2/M for a total of k^5/M);
- every n-body vertex with $2i$ derivatives: $k^{2i}R^{2i+3n-5}/M$. (Here R is the generic underlying length scale or inverse momentum.) The result is that a diagram with E external lines and V_{2i}^n vertices scales as k^ν with

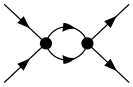
$$\nu = 5 - \frac{3}{2}E + \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(2i+3n-5)V_{2i}^n \ . \tag{21}$$

The application of these rules to the leading diagrams is in Fig. 4.

Let's verify this formula for a few cases. Start with one two-body vertex with no derivatives, so $V_0^2 = 1$ ($n = 2, i = 0$); four external lines so $E = 4$, yielding

$$\nu = 5 - \frac{3}{2}\cdot 4 + (2\cdot 0 + 3\cdot 2 - 5)\cdot 1 = 0 \quad \Longrightarrow \quad k^0 \tag{22}$$

Try a diagram with a loop: two 2-body vertices with no derivatives, so $V_0^2 = 2$ ($n = 2, i = 0$), one loop, so $L = 1$, 4 external lines



$$\nu = 5 - \frac{3}{2}\cdot 4 + (2\cdot 0 + 3\cdot 2 - 5)\cdot 2 = 1 \quad \Longrightarrow \quad k^1 \tag{23}$$

Finally, try a vertex with two derivatives, so $V_2^2 = 1$ ($n = 2$,$i = 1$), yielding

$$\nu = 5 - \frac{3}{2}\cdot 4 + (2\cdot 1 + 3\cdot 2 - 5)\cdot 1 = 2 \quad \Longrightarrow \quad k^2 \ . \tag{24}$$

So the formula seems to work (try the 3-body vertex!). A consequence is that only a finite number of diagrams (which can be reverse-enineered from the corresponding terms) contribute to a given order in k, which means to a given order in the kR expansion.

Matching (to the underlying theory *or* to data) yields:

$$C_0 = \frac{4\pi}{m}a_0 = \frac{4\pi}{m}R \ , \quad C_2 = \frac{4\pi}{m}\frac{a_0^2 r_0}{2} = \frac{4\pi}{m}\frac{R^3}{3} \ , \quad C_2' = \frac{4\pi}{m}a_p^3 \quad \cdots \tag{25}$$

In general, we see that dimensional analysis applied to either Eq. (8) or Eq. (9) tells us that

$$C_{2i} \sim \frac{4\pi}{m}R^{2i+1} \ , \quad D_{2i} \sim \frac{4\pi}{m}R^{2i+4} \tag{26}$$

Every term comes with the same $1/m$ factor (this is from Galilean invariance; relativistic corrections would give higher powers of $1/m$). The factor of 4π doesn't come from dimensional analysis (since it is dimensionless!) but is manifest from connecting the scattering amplitude to the T-matrix.

b.6 Recipe for an effective field theory

We can summarize the steps that we've applied for a particular simple EFT that will also apply more generally:

1. Use the most general Lagrangian with low-energy degrees of freedom consistent with global and local symmetries of underlying theory
   - For purely short-distance interactions, the Lagrangian
   $$\mathcal{L}_{\rm eft} = \psi^\dagger \big[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M} \big] \psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \dots \,, \tag{27}$$
   is general (but not unique).
2. Declaration of a regularization and renormalization scheme
   - We need to add a regulator that makes the theory well behaved (but wrong!) at high energies (high momenta in our loop integrals, which are sums over intermediate states).
   - For a natural-sized scattering length a_0, dimensional regularization and minimal subtraction are the most efficient but one can use a sharp cutoff as well as other functions that are smoother (exercise!).
3. Identify a well-defined power counting associated with a (small) expansion parameter
   - Use the separation of scales to form small ratios, which in the case just considered is $\frac{k}{\Lambda}$ with $\Lambda \sim 1/R \Longrightarrow ka_0 \ll 1$, etc.
   - This recovers the effective range expansion order-by-order with diagrams
   $$f_0(k) \propto \frac{-1}{k\cot\delta_0(k) - ik} \longrightarrow a_0\big[1 - ia_0k - (a_0^2 - a_0r_0/2)k^2 + \mathcal{O}(k^3a_0^3)\big] \tag{28}$$
   $$\longrightarrow R[1 - ikR - 2k^2R^2/3 + \mathcal{O}(k^3R^3)] \quad \text{[hard sphere]} \tag{29}$$
     - With DR/MS, there is one power of k per diagram, *natural* coefficients.
     - We can estimate the *truncation error* from (naive) dimensional analysis if we assume that the coefficient is close to one.
     - We know when it breaks down (when k gets close to the underlying physics scale).
     - This is valid for *any* natural short-range interaction! (That is, not just for our example of hard-sphere scattering.)

In the next section, we will extend the discussion to the interesting case of large scattering lengths and then we'll elaborate on the discussion here.

c. Adding spin (see also Section 7)

What if we have spin dependence? If we restrict ourselves to central forces, from the discussion of general potentials we can have a term in the effective potential proportional to $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$. So it would

seem that we need to introduce another independent coupling besides the C_0. But it turns out not to be independent (if all we have are neutrons)!

Because the spatial wave function must be symmetric to have a non-zero amplitude at the same $\mathbf{x}$, one must be spin up and one spin down in an anti-symmetric combination, i.e., in a ${}^1\mathrm{S}_0$ state. This means that of the two possible S-wave partial waves ${}^1\mathrm{S}_0$ and ${}^3\mathrm{S}_1$ that could have couplings (or scattering lengths) associated with them, only one of them is non-zero. In the exercises there is a problem that applies *Fierz rearrangement* methods to show that

$$(\psi^\dagger \boldsymbol{\sigma} \psi) \cdot (\psi^\dagger \boldsymbol{\sigma} \psi) = -3(\psi^\dagger \psi)^2 \,. \tag{30}$$

Therefore

$$C_S(\psi^\dagger \psi)^2 + C_T(\psi^\dagger \boldsymbol{\sigma} \psi)^2 = (C_S - 3C_T)(\psi^\dagger \psi)^2 \,, \tag{31}$$

and we can eliminate the C_T term entirely. (Or we can eliminate the C_S term! We'll do this in a future section.)

Further discussion is given in Section 7.

d. Introduction to 3-body forces from "integrating out" dofs

d.1 Classical analogy: tidal forces

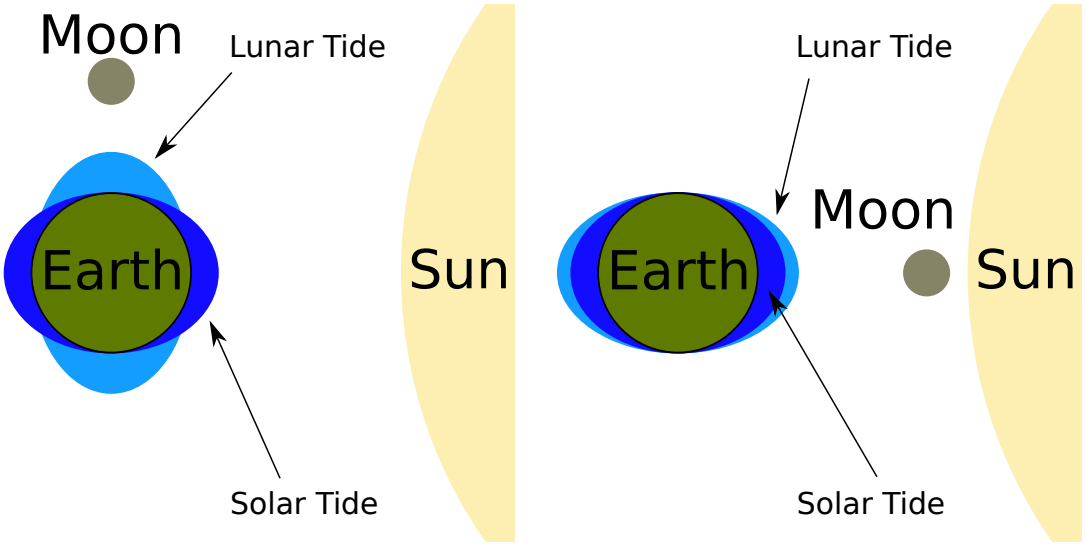


Figure 5: Analog of quantum three-body forces from tidal forces. [credit: K. Wendt]

Why do we have three-body forces in a low-energy effective theory? Let's start by considering a classical analog of describing the interaction of the sun, moon, and earth by gravity. Our low-energy theory replaces these finite composite systems by point masses at the center-of-mass of each body. This gives us a first approximation to energy of the system. But if we want to be more

accurate, we have to recognize that because of tidal forces, the gravitational force on the Earth is not just the pairwise sum of the point-like Earth-Moon and Earth-Sun forces. Because the Earth is actually a composite object, it gets "polarized" and the force between the Earth and the Sun in Fig. 5 depends on the location of the Moon. Thus to take this into account we need a force that depends on the coordinates of all three bodies; in our low-energy theory of point masses, this is a three-body force between them.

d.2 Atomic three-body forces

This would seem to be a general phenomenon that we would expect when we describe any composite particles as point particles in a low-energy theory. In particular, at the quantum mechanics level we can think about the forces between atoms in such a theory. You'll recall that there is an effective potential between two atoms from the mutual polarization, which leads to the intermediate-range van der Waals force, which is described (for example) by the familiar Lennard-Jones potential where the atoms are represented as point particles (with the potential being a function of the distance between the center-of-masses of the atoms). Note that the qualitative form of the potential is the same as the local nuclear potential: attraction at intermediate distances and strong repulsion at short distances (in the case of atoms from the overlap of electron clouds).

So now one would expect from the analogy to the Earth-Moon-Sun system that the force between two atoms would be modified by the polarization due to a third atom. This is very natural, but we don't seem to be taught about such three-body forces! However, they are real and in fact were described by Axilrod and Teller way back in 1943. To be specific, the three-body potential for atoms or molecules is from triple-dipole mutual polarization and arises as a 3rd-order perturbation theory correction (cf. dipole-dipole mutual polarization at 2nd order).

Referring to Fig. 6, the three-body potential is

$$V(i,j,k) = \frac{\nu(1+3\cos\theta_i\cos\theta_j\cos\theta_k)}{(r_{ij}r_{ik}r_{jk})^3} \ , \tag{32}$$

although the specific details are not important for us. Some comments on this three-body force:

- It is usually negligible in metals and semiconductors because the fine-structure constant is so small. So we rarely hear about it.
- However, it can be important for the ground-state energy of solids bound by van der Waals potentials.
- It is used to improve the accuracy of calculations performed on Van der Waals clusters such as those formed by the noble gases.
- Bell and Zuker (1976) find it to be 10% of energy in solid xenon. (Note: the typical size of the three-body contribution to the triton binding energy is also about 10%!)
- See "Local Density Theory of Polarizability" by Mahan and Subbaswamy for more details.

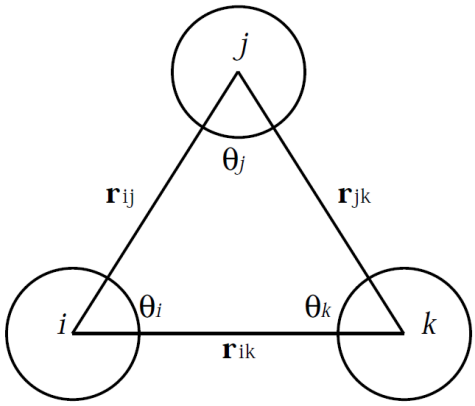


Figure 6: Diagram defining the angles and distance for the Axilrod-Teller three-body force.

We can say more generally that these three-body forces arise from the elimination of degrees of freedom in our low-energy theory. If we included the positions of all the masses in the Earth-Moon-Sun or the electrons (and nuclei) in the atoms, we would get the result from just summing two-body forces. But by eliminating those variables (degrees of freedom) in favor of collective coordinates (center-of-mass position), we introduce three-body forces.

d.3 Preview of nuclear three-body forces

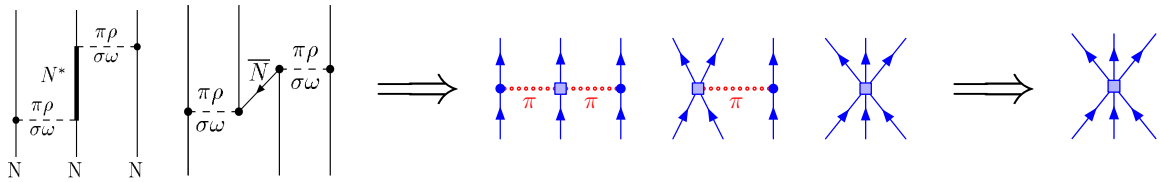


Figure 7: Sources of three-body forces and the diagrams for their contributions in chiral effective field theory and pionless EFT.

Now consider a system of three (or more nucleons). In our low-energy effective theory we have also eliminated degrees of freedom. Some examples are shown in Fig. 7):

- excited states of nucleon, generically denoted N^* in the figure;
- relativistic effects (eliminating anti-particles);
- integrating out high-momentum intermediate states (next week!).

When we omit these contributions, the effects of the heavy meson exchanges are shrunk to points and included in a derivative expansion, as in our example at the beginning. If the pion is treated as a heavy degree of freedom, it will also shrink and we will only have diagrams like the contact interaction on the right (but with vertices with more and more powers of momentum or more and more derivatives, as in the two-body case). If we resolve the pion, we will have mid-range (middle),

and long-range (left) three-body forces, which will have characteristic spin- and isospin-dependence because of the pion-nucleon interaction. We will discuss the impact of such forces on nuclear structure in later sections.

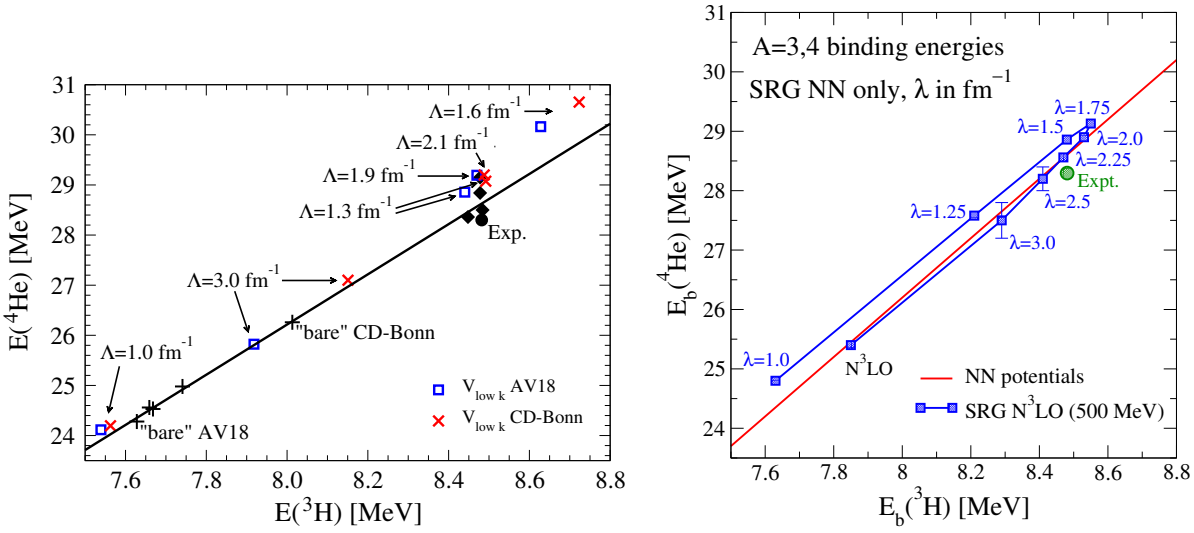


Figure 8: Tjon line: correlation of alpha particle and triton binding energies with different potentials. Note where experiment lies. The points on the left marked with Λ values are one type of renormalization group (RG) evolved potential, called $V_{\mathrm{low}\,k}$. The points on the right marked with λ values are a different type of RG-evolved potential, called SRG. More about these later!

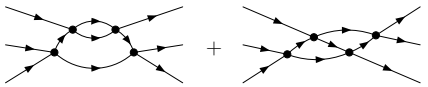
For now, we can ask for phenomenological evidence that we will need three-body forces. To do so we can turn to the set of high-precision NN interactions we discussed in an earlier section. These describe the two-body physics (nucleon-nucleon scattering and the deuteron) essentially perfectly up to a rather high energy. But now that us take all of the ones available and calculate accurately the binding energies of the triton (^{3}H) and the alpha particle (^{4}He). If we plot the results and also the experimental point, we find that none of the potentials agree with experiment, but interestingly they all lie close to a line, which is called the Tjon line (see Fig. 8). To agree with experiment, for each potential we need to add a three-body force and the energy we need is generally different in each case. What about our expectations for 4-body and higher body?

d.4 Three-body forces in pionless EFT

Finally, let's return to the pionless EFT and some more formal considerations. Feynman rules and power counting predict a three-body force is possible and because it is allowed by the symmetries, general EFT principles say it *will* appear. If we have only neutrons, the value is zero from the Pauli principle because we can't have a non-zero wave function for all three particle at the same point $\mathbf{x}$ since at least two spins must be the same.

But when we have protons and neutrons, our simple perturbative case tells us we *need* a three-body contact force to fix the contribution from our two-body interactions at high energy when we

consider the scattering of three nucleons. At low resolution, we don't resolve a series of two-body scatterings at high energy, and this implies that we need three-body, even if the underlying potential we are trying to reproduce (e.g., hard-sphere scattering) is two-body only! This shows up as new *logarithmic* divergences in 3–3 scattering in these diagrams:

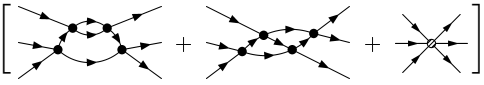


$$\propto (C_0)^4 \ln(k/\Lambda_c) \tag{33}$$

That is, if you evaluate the contribution of these Feynman diagrams, you will find that it contains a dependence on the momentum cutoff as indicated. The changes in Λ_c *must* be absorbed by a *3-body* coupling $D_0(\Lambda_c)$ where

$$D_0(\Lambda_c) \propto (C_0)^4 \ln(a_0\Lambda_c) + \text{const.} \quad \text{[from Braaten \& Nieto]} \tag{34}$$

. The requirement that the total (which is part of a measurable quantity) be independent of Λ_c (which is an auxiliary parameter that must disappear in the final result) tells us that any change with Λ_c in the diagrams must be compensated by $D_0(\Lambda_c)$:

$$\frac{d}{d\Lambda_c}$$  $$= 0 \,, \tag{35}$$

and this fixes the coefficient! This is an example of using the *renormalization group* running of a coupling to derive new information, seemingly at no cost!