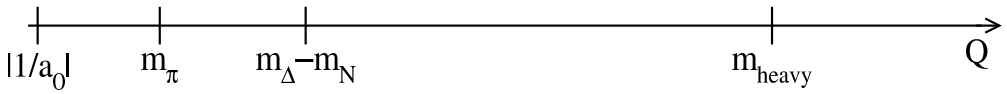
7. Renormalization and universality in pionless EFT

Recall the scales of nuclear forces from Section 5: Pionless EFT is applicable to the region $Q \ll m_\pi$.



This EFT offers a systematic expansion in $Q/\Lambda_{\rm breakdown}$, where the breakdown scale should be of order m_π. We've already seen in Section 6 how we can describe the scattering amplitude (or on-shell T-matrix) in perturbation theory in kR (reproducing the effective range expansion) if the theory is natural. In practice this means all of the ERE coefficients, and a_0 in particular, are of order R. We first consider an explicit example of matching for the perturbative case and then consider the *unnatural* case.

a. Perturbative matching to a (toy) underlying theory

This section is a slightly edited excerpt from "Perturbative Effective Field Theory at Finite Density" by Furnstahl, Steele, and Tirfessa, nucl-th/9910048 or Nucl. Phys. A **671**, 396 (2000) [4].

As a concrete example of perturbative matching an underlying theory to a low-energy EFT, we take the underlying potential to be a separable potential in the ${}^1{\rm S}_0$-channel[1] that falls off at large momenta and depends on the relative momentum $k \equiv |\mathbf{k}| \equiv \frac{1}{2}|\mathbf{k_1} - \mathbf{k_2}|$ only,

$$\langle \mathbf{k}'|\hat{V}_{\rm true}|\mathbf{k}\rangle = \frac{4\pi}{M}\frac{\alpha m^3}{(k^2+m^2)(k'^2+m^2)}\,. \tag{1}$$

[Recall that Galilean boost invariance requires the interaction between two free nucleons of momentum $\mathbf{k_1}$ and $\mathbf{k_2}$ to be independent of their center-of-mass momentum $\mathbf{P} = \mathbf{k_1} + \mathbf{k_2}$.] The mass m corresponds to the range (and non-locality) of the potential; it plays the role of the underlying short-distance scale. We adopt an overall normalization of $4\pi/M$, with M the nucleon mass. The dimensionless coupling α provides a perturbative expansion parameter. If α is $\mathcal{O}(1)$, then the effective range parameters are of "natural" size, which means a constant of order one times, in this case, a power of m (details in the Appendix of [4]).

An EFT can describe observables from the interaction in Eq. (1) to any desired accuracy as long as the details of the underlying potential are not probed. For scattering observables, this restricts the external momentum k to be much less than the characteristic mass m. The effective potential can then be written as a momentum expansion

$$\langle \mathbf{k}'|\hat{V}_{\rm EFT}|\mathbf{k}\rangle = C_0 + C_2\frac{\left(k^2+k'^2\right)}{2} + C_4\frac{\left(k^2+k'^2\right)^2}{4} + \widetilde{C}_4\frac{\left(k^2-k'^2\right)^2}{4} + \dots\,, \tag{2}$$

which must be regulated. Two possible regularization schemes are dimensional regularization with power divergence subtraction (DR/PDS) [8] and cutoff regularization (CR). In the latter case we use

[1] An extension to higher partial waves is straightforward and does not introduce new features into the analysis.

a gaussian separable cutoff in momentum space that strongly damps momenta above an arbitrary cutoff Λ_c:

$$\langle \mathbf{k}'|\hat{V}_{\rm EFT}|\mathbf{k}\rangle = \left[C_0 + C_2\frac{\left(k^2+k'^2\right)}{2} + C_4\frac{\left(k^2+k'^2\right)^2}{4} + \widetilde{C}_4\frac{\left(k^2-k'^2\right)^2}{4} + \ldots\right] e^{-(k^2+k'^2)/2\Lambda_c^2} \ . \quad (3)$$

For perturbative calculations, analytic expansions can be obtained for these schemes in both free-space and in a uniform finite-density system.

The idea of a perturbative matching calculation is that we can work to $n^{\rm th}$ order in the momentum expansion and determine the corresponding coefficients $C_0, \ldots, C_{2n}$ order-by-order perturbatively in the coupling α by equating observables calculated from the "true" and EFT hamiltonians [10]. We define the series expansion coefficients $C_{2n}^{(s)}$ by

$$C_{2n} = \sum_{s=1}^{\infty} \alpha^s \, C_{2n}^{(s)} \ . \quad (4)$$

It is not necessary that perturbation theory for the observables converges in the low-momentum limit,[2] but for our example it does. We will carry out the matching for C_0 in some detail so that the generalization to higher orders and the extension to finite density is clear. We choose the on-shell T-matrix as our matching observable, since it has a natural perturbative expansion: the Born series $\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \cdots$, with $\hat{G}_0 = (E - \hat{H}_0)^{-1}$.

We start our discussion with cutoff regularization because the physical content of the renormalization is manifest. At $\mathcal{O}(\alpha)$, the on-shell T-matrix is just the on-shell potential $\hat{V}$, which for $k \ll m, \Lambda_c$ can be expanded (it is sufficient to take $\mathbf{k}' = \mathbf{k}$ for S-waves):

$$\langle \mathbf{k}|\hat{V}|\mathbf{k}\rangle = \begin{cases} \dfrac{4\pi}{M}\dfrac{\alpha m^3}{(k^2+m^2)^2} = \dfrac{4\pi\alpha}{Mm} + \mathcal{O}(k^2), & \text{true} \\ \alpha C_0^{(1)} e^{-k^2/\Lambda_c^2} = \alpha C_0^{(1)} + \mathcal{O}(k^2) \ , & \text{EFT (CR)} \end{cases} \quad (5)$$

where we have used the shorthand notation $\mathcal{O}(k^2)$ to denote natural corrections with the proper dimension multiplied by either k^2/m^2 or k^2/Λ_c^2 as applicable. Matching to this order fixes

$$C_0^{(1)} = \frac{4\pi}{Mm} \ , \quad (6)$$

and is indicated schematically in Fig. 1a.

Matching at $\mathcal{O}(\alpha^2)$ is illustrated in Fig. 1b. There are two contributions on the EFT side: $\langle \hat{V}\hat{G}_0\hat{V}\rangle$ evaluated using $C_0^{(1)}$ fixed by Eq. (6), and $\langle \hat{V}\rangle$ evaluated with the $\mathcal{O}(\alpha^2)$ contribution to

[2]The main assumption is that the short-distance physics is perturbative. The coefficients will have convergent expansions in α since, as we will see explicitly, higher-order constants incorporate high momentum physics only and so are not sensitive to possible infrared divergences (e.g., from a long-range Coulomb potential). This means that perturbative matching is sufficient even if the constants will then be used in a nonperturbative calculation (see Ref. [10] for an example).

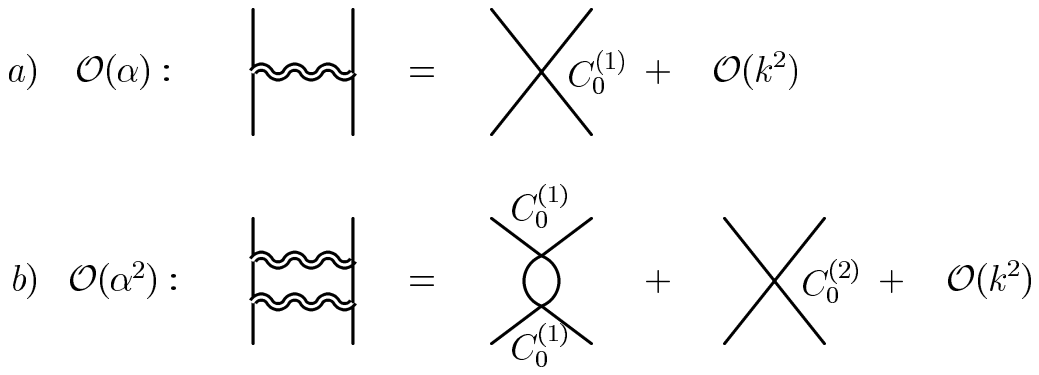


Figure 1: Perturbative matching in free-space to $\mathcal{O}(k^2)$. The double line for the potential is a reminder that the potential is separable.

C_0, namely, $C_0^{(2)}$. The latter *renormalizes* C_0. To illustrate the physics of this renormalization, we consider explicitly the difference of $\langle \hat{V}\hat{G}_0\hat{V}\rangle_{\rm true}$ and $\langle \hat{V}\hat{G}_0\hat{V}\rangle_{\rm EFT}$:

$$\Delta\langle \hat{V}\hat{G}_0\hat{V}\rangle = \left(\frac{4\pi\alpha}{Mm}\right)^2 \left\{ \frac{m^2}{k^2+m^2}\left[\int \frac{d^3q}{(2\pi)^3}\,\frac{m^2}{q^2+m^2}\,\frac{M}{k^2-q^2+i\eta}\,\frac{m^2}{q^2+m^2}\right]\frac{m^2}{k^2+m^2} \right.$$
$$\left. - e^{-k^2/2\Lambda_c^2}\left[\int \frac{d^3q}{(2\pi)^3}\,e^{-q^2/2\Lambda_c^2}\,\frac{M}{k^2-q^2+i\eta}\,e^{-q^2/2\Lambda_c^2}\right] e^{-k^2/2\Lambda_c^2} \right\} . \tag{7}$$

Because we are only working to $\mathcal{O}(k^2)$ in the effective potential, we can expand all k^2 dependence except for $\hat{G}_0$, since q can be smaller than k:

$$\Delta\langle \hat{V}\hat{G}_0\hat{V}\rangle = \left(\frac{4\pi\alpha}{Mm}\right)^2 \int_0^\infty \frac{dq\,q^2}{2\pi^2}\,\frac{M}{k^2-q^2+i\eta}\left[\frac{m^4}{(q^2+m^2)^2} - e^{-q^2/\Lambda_c^2}\right] + \mathcal{O}(k^2)$$
$$= -\frac{4\pi\alpha^2}{Mm}\left(\frac{1}{2} - \frac{1}{\sqrt{\pi}}\frac{\Lambda_c}{m}\right) + \mathcal{O}(k^2)\ . \tag{8}$$

The constant $C_0^{(2)}$ is chosen to cancel the k^2–independent part of $\Delta\langle \hat{V}\hat{G}_0\hat{V}\rangle$, so that the net result for $\langle \hat{V} + \hat{V}\hat{G}_0\hat{V}\rangle$ is $\mathcal{O}(k^2)$:

$$C_0^{(2)} = -\frac{4\pi}{Mm}\left(\frac{1}{2} - \frac{1}{\sqrt{\pi}}\frac{\Lambda_c}{m}\right)\ . \tag{9}$$

Choosing the cutoff $\Lambda_c \approx m$ keeps the constants natural and gives the maximum range of validity for the EFT [11].

As expected from the uncertainty principle, the local interaction $C_0^{(2)}$ is determined entirely by high-momentum ($q \lesssim \Lambda_c$) contributions in the loop integral [10]. This is because for $q \ll \Lambda_c$, the true and EFT tree-level results have already been matched in Eq. (5) to leading order in q^2 and therefore cancel in the integrand of Eq. (8). Note that $C_0^{(2)}$ removes the Λ_c dependence while correcting for the contributions in the loop integral from the high-energy states. The ability

to absorb the high-momentum components of the interaction into the constants of the effective potential is a generic feature of an EFT approach and is essential for a systematic prediction.

If we add a long-range potential to the true theory, we must also reproduce its long-wavelength effects in the EFT, so we will still have agreement in the low-momentum region where $q \ll \Lambda_c$, and the analysis goes through unchanged. If we extend the effective potential to include two constants, the $\alpha C_2^{(1)} q^2$ piece serves to make the low-momentum part of the loop integrals agree to $\mathcal{O}(q^4)$, and the high-momentum part of $\Delta\langle \hat{V}\hat{G}_0\hat{V}\rangle$ is absorbed into $\alpha^2(C_0^{(2)} + C_2^{(2)}k^2)$. As a result, the EFT reproduces the observables to $\mathcal{O}(k^4)$. The addition of C_2 requires an additional renormalization of C_0 in cutoff regularization (see Appendix of [4]), which means that all the constants change with each successive order in the momentum expansion.

Having convinced ourselves of the physical origin of the constants, we move to the simpler but physically less transparent dimensional regularization. It has been shown that power counting with only short-range potentials is regularization scheme independent for free-space observables [11, 12], as considered here. Comparing the expressions for the true and effective T-matrix to leading order in α again leads to

$$\langle \mathbf{k}|\hat{V}|\mathbf{k}\rangle = \begin{cases} \dfrac{4\pi}{M}\dfrac{\alpha m^3}{(k^2+m^2)^2} = \dfrac{4\pi\alpha}{Mm} + \mathcal{O}(k^2)\,, & \text{true} \\ \alpha C_0^{(1)} + \mathcal{O}(k^2)\,, & \text{EFT (PDS)} \end{cases} \tag{10}$$

which fixes the first term in Eq. (4) to be

$$C_0^{(1)} = \frac{4\pi}{Mm}\ , \tag{11}$$

just as with the cutoff regulator.

At the next order in α, the difference between $\langle \hat{V}\hat{G}_0\hat{V}\rangle_{\text{true}}$ and $\langle \hat{V}\hat{G}_0\hat{V}\rangle_{\text{EFT}}$ becomes

$$\Delta\langle \hat{V}\hat{G}_0\hat{V}\rangle = \left(\frac{4\pi\alpha}{Mm}\right)^2 \left\{ \frac{m^2}{k^2+m^2}\left[\int \frac{d^3q}{(2\pi)^3}\,\frac{m^2}{q^2+m^2}\,\frac{M}{k^2-q^2+i\eta}\,\frac{m^2}{q^2+m^2}\right]\frac{m^2}{k^2+m^2} \right.$$
$$\left. - \left[\left(\frac{\mu}{2}\right)^{4-D}\int \frac{d^{D-1}q}{(2\pi)^{D-1}}\,\frac{M}{k^2-q^2+i\eta}\right]\right\}\,, \tag{12}$$

where the second integral is dimensionally regularized with $D = 4+\epsilon$ and in general is evaluated in the PDS scheme as [8]

$$\left(\frac{\mu}{2}\right)^{4-D}\int \frac{d^{D-1}q}{(2\pi)^{D-1}}\,\frac{Mk^{2i}q^{2j}}{k^2-q^2+i\eta} = -\frac{Mk^{2(i+j)}}{4\pi}(\mu+ik)\ . \tag{13}$$

Here μ is the DR renormalization scale. As before, the low-momentum parts of the integrands already agree to $\mathcal{O}(q^2)$, so the constant difference is from the high-momentum behavior.

The results for the first two terms in the Born series are

$$\langle \mathbf{k}|\hat{V}+\hat{V}\hat{G}_0\hat{V}|\mathbf{k}\rangle = \begin{cases} \dfrac{4\pi}{M}\left[\dfrac{\alpha m^3}{(k^2+m^2)^2}+\dfrac{\alpha^2 m^6}{(k^2+m^2)^4}\left(\dfrac{k^2-m^2}{2m}-ik\right)\right], & \text{true} \\ \alpha C_0^{(1)}+\alpha^2 C_0^{(2)}-\dfrac{\alpha^2 M\left(C_0^{(1)}\right)^2}{4\pi}(\mu+ik), & \text{EFT (PDS)} \end{cases} \tag{14}$$

which requires

$$C_0^{(2)} = -\frac{Mm}{4\pi}\left(C_0^{(1)}\right)^2\left(\frac{1}{2}-\frac{\mu}{m}\right) = -\left(\frac{4\pi}{Mm}\right)\left(\frac{1}{2}-\frac{\mu}{m}\right), \tag{15}$$

for a proper match to $\mathcal{O}(\alpha^2, k^2)$. Note that for $\Lambda_c = \sqrt{\pi}\mu$, the DR/PDS constants are equivalent to the cutoff results Eqs. (6) and (9). This agreement does not persist at higher orders in momentum.

Carrying the matching to one more order in α suggests a geometric series for the $C_0^{(s)}$ in the DR/PDS scheme, which indeed sums to the full nonperturbative solution (see Appendix of [4]):

$$C_0 = \frac{4\pi\alpha}{Mm}\left[1+\alpha\left(\frac{1}{2}-\frac{\mu}{m}\right)\right]^{-1}. \tag{16}$$

Extending the analysis to higher orders in momentum (or expanding the full results from the Appendix of [4] in powers of α) gives

$$C_2^{(1)} = -\left(\frac{4\pi}{Mm}\right)\frac{2}{m^2}, \qquad C_2^{(2)} = \left(\frac{4\pi}{Mm}\right)\left(\frac{5}{8}-\frac{\mu}{m}\right)\frac{4}{m^2}, \tag{17}$$

$$C_4^{(1)} = \left(\frac{4\pi}{Mm}\right)\frac{3}{m^4}, \qquad C_4^{(2)} = -\left(\frac{4\pi}{Mm}\right)\left(\frac{7}{10}-\frac{\mu}{m}\right)\frac{10}{m^4}. \tag{18}$$

As additional constants are added to the effective potential in DR/PDS, the previously fixed constants are not modified. This is not the case in CR (see Appendix of [4]) due to terms proportional to positive powers of the cutoff.

The constant $\widetilde{C}_4$ requires additional input, because it does not contribute to the on-shell two-body T-matrix. It is tempting to match the true and EFT T-matrices *off shell* to determine $\widetilde{C}_4$, but this is never necessary, as only observables are required to fix the EFT constants. In particular, the additional *on-shell* constraint of matching the three-body scattering amplitudes [2, 1] can be used to find $\widetilde{C}_4$. Because we are in a regime where the coupling is perturbative, the Faddeev equations are simplified to a set of three-to-three tree-level amplitudes at second-order in the potential, $\mathcal{O}(\alpha^2)$. The result from matching is:

$$\widetilde{C}_4^{(1)} = \left(\frac{4\pi}{Mm}\right)\frac{1}{m^4}. \tag{19}$$

The second-order constant $\widetilde{C}_4^{(2)}$ as well as true three-body terms (needed to absorb the divergences in three-to-three loop-level amplitudes) do not enter nuclear matter calculations until higher order in the α expansion.

[Note from Ref. [5], which is about field redefinitions: ... we can arbitrarily trade-off the value of $\widetilde{C}_4$ with the coefficient of a three-body contact term. This means that off-shell matrix elements of the potential (e.g., $\langle \mathbf{0}|\hat{V}_{\rm EFT}|\mathbf{k}\rangle$) and subsequently the off-shell T-matrix can be changed continuously by varying α, without changing *any* observables. One choice is to eliminate two-body off-shell vertices such as $\widetilde{C}_4$ entirely in favor of many-body vertices that do not vanish on shell. This is the form of many-body EFT used in Ref. [7], which corresponds to Georgi's "on-shell effective field theory" [6]. In different situations, different choices may be more efficient [3].]

By construction, the EFT systematically reduces the error order-by-order in the momentum expansion. We can see this explicitly by examining the truncation error in $\langle \hat{T}\rangle$ to $\mathcal{O}(\alpha^2)$, which is just $\langle \hat{V} + \hat{V}\hat{G}_0\hat{V}\rangle$, between calculations using the true and EFT potentials.[3] For an effective potential only containing C_0, the error is $\mathcal{O}(k^2)$

$$\Delta\langle \hat{V} + \hat{V}\hat{G}_0\hat{V}\rangle = \frac{4\pi}{Mm}\begin{cases} \left[\alpha\left(-2+\dfrac{m^2}{\Lambda_c^2}\right)+\alpha^2\left(\dfrac{5}{2}-\dfrac{m^2}{2\Lambda_c^2}-\dfrac{2m}{\sqrt{\pi}\Lambda_c}\right)\right]\left(\dfrac{k}{m}\right)^2 & \text{CR} \\ \left(-2\alpha+\dfrac{5}{2}\alpha^2\right)\left(\dfrac{k}{m}\right)^2 & \text{DR/PDS} \end{cases} + \mathcal{O}(k^3)\ , \tag{20}$$

and adding C_2 brings the error to $\mathcal{O}(k^4)$

$$\Delta\langle \hat{V} + \hat{V}\hat{G}_0\hat{V}\rangle = \frac{4\pi}{Mm}\begin{cases} \left\{\alpha\left(3-\dfrac{2m^2}{\Lambda_c^2}+\dfrac{m^4}{2\Lambda_c^4}\right)+\alpha^2\left[-7+\dfrac{5m^2}{2\Lambda_c^2}-\dfrac{m^4}{4\Lambda_c^4}\right.\right. \\ \qquad \left.\left.+\dfrac{4\Lambda_c}{\sqrt{\pi}m}\left(1+\dfrac{m^2}{\Lambda_c^2}-\dfrac{5m^4}{12\Lambda_c^4}\right)\right]\right\}\left(\dfrac{k}{m}\right)^4 & \text{CR} \\ \left(3\alpha-7\alpha^2\right)\left(\dfrac{k}{m}\right)^4 & \text{DR/PDS} \end{cases} + \mathcal{O}(k^5)\ . \tag{21}$$

Note that the truncation error with CR depends on Λ_c while in the DR/PDS scheme the truncation error is independent of μ. Extending the analysis to nonperturbative calculations and finite density will in general require numerical solutions, and so a connection between the analytical results above and a graphical error analysis is important. The error plots introduced by Lepage [10] are useful in this regard.

Keeping constants to $\mathcal{O}(k^0)$, $\mathcal{O}(k^2)$, and $\mathcal{O}(k^4)$ in the effective potential Eq. (2) or Eq. (3) leads to successively better approximations to $\langle \hat{V} + \hat{V}\hat{G}_0\hat{V}\rangle$, as seen by the analytic expressions for the error Eqs. (20) and (21) and shown graphically in Fig. 2. With each additional order, the slope of the error increases, reflecting the improved truncation error. (If a long-range potential is added to both the true and effective potentials, the absolute error in the DR/PDS power-counting scheme

[3] For the present perturbative case, this is more convenient than looking at the error in $k\cot\delta$, which is appropriate for a nonperturbative calculation. In the error plots, we consider only the real part of the difference in $\langle \hat{V} + \hat{V}\hat{G}_0\hat{V}\rangle$. The imaginary part follows from unitarity, which the EFT reproduces order-by-order.

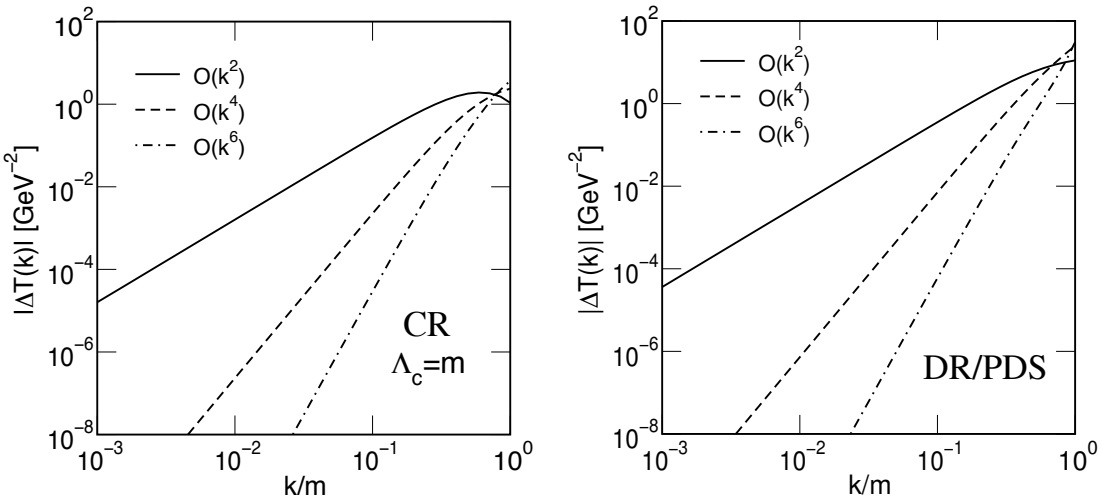


Figure 2: $|\Delta\,\mathrm{Re}\langle\mathbf{k}|\hat{T}|\mathbf{k}\rangle|$ to $\mathcal{O}(\alpha^2)$ vs. k/m for $\alpha = -1/2$ and with $M = 940$ MeV and $m = 600$ MeV. Results for cutoff regularization (CR) with $\Lambda_c = m$ are shown on the left and for dimensional regularization (DR/PDS) on the right. Note that $a_s = -2/3m$ and $r_e = 9/m$.

will decrease but the error is always $\mathcal{O}(k^2)$ [11, 9].) For $k \ll m$ the error is dominated by the first unmatched term, so the lines are straight on a log-log plot. The EFT should break down when the external momentum probes the details of the underlying potential; graphically the intersection of the lines[4] indicates the approximate breakdown scale Λ. In our example, we see that $\Lambda \approx m$, as expected for a natural theory [with $\alpha \approx \mathcal{O}(1)$]; indeed, all the errors are of the same order for $k \approx m$. This breakdown scale does not change as more orders in α are included in the contact interactions, but the accuracy does improve.

b. Non-perturbative matching

First we consider the leading order coefficient C_0 for the unnatural case of large scattering length. That means we expand about the $|1/a_0| = 0$ limit (so instead of $ka_0 \ll 1$, we have $ka_0 \gg 1$). The plan is to match the expansion of the Lippmann-Schwinger (LS) equation for the T-matrix to the on-shell form for the T-matrix from the effective range expansion:

$$T = \frac{4\pi}{m}\frac{1}{\frac{1}{a_0} - \frac{1}{2}r_0 + ik}\,. \tag{22}$$

We can do non-perturbative matching because we can solve the Lippmann-Schwinger (LS)

[4] To determine the intersection scale, one should extend the lines from the straight regions ($k \ll m$) rather than looking at the actual intersection.

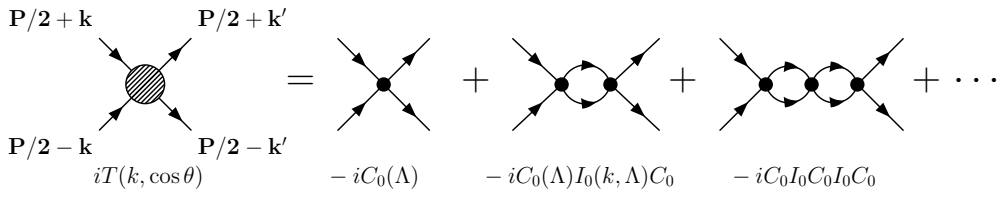


Figure 3: Lippmann-Schwinger equation for the T matrix.

equation exactly for $V = C_0$:

$$T = C_0 + C_0 I_0(k,\Lambda_c) C_0 + C_0 I_0(k,\Lambda_c) C_0 I_0(k,\Lambda_c) C_0 + \cdots \tag{23}$$

$$= C_0[1 + I_0(k,\Lambda_c) C_0 + (I_0(k,\Lambda_c) C_0)^2 + \cdots] \tag{24}$$

$$= \frac{C_0}{1 - C_0 I_0(k,\Lambda_c)} = \frac{1}{\frac{1}{C_0} - I_0(k,\Lambda_c)} \tag{25}$$

where

$$I_0(k,\Lambda_c) = -\frac{m}{4\pi}\left(ik + \frac{2}{\pi}\Lambda_c - \mathcal{O}(k^2/\Lambda_c^2)\right) . \tag{26}$$

If we take Λ_c large, then the $\mathcal{O}(k^2/\Lambda_c^2)$ correction is small. For other ways to regulate the integral with a cutoff, Λ_c will appear in each case, but the numerical coefficient is different.

Matching to the scattering length only by equating (22) and (25):

$$\frac{4\pi}{m}\frac{1}{\frac{1}{a_0} + ik} = \frac{1}{\frac{1}{C_0} - I_0(k,\Lambda_c)} = \frac{1}{\frac{1}{C_0} + \frac{m}{4\pi}(ik + \frac{2}{\pi}\Lambda_c + \mathcal{O}(k^2/\Lambda_c^2))} , \tag{27}$$

which yields

$$\frac{1}{a_0} = \frac{4\pi}{C_0} + \frac{2}{\pi}\Lambda_c \tag{28}$$

or

$$C_0(\Lambda_c) = \frac{4\pi}{m}\frac{1}{\frac{1}{a_0} - \frac{2}{\pi}\Lambda_c} . \tag{29}$$

This *running coupling* C_0 will give cutoff independent results at low k (up to $\mathcal{O}(k^2/\Lambda_c^2)$ corrections).

So at leading order, we have to resum all of the C_0 interactions. Beyond leading order, we will still have to resum the C_0 interactions whenever they occur, but then treat higher-order two-body interactions perturbatively. Three-body interactions are another story, however (stay tuned)!

Let's discuss the behavior of $C(\Lambda_c)$ in the strong and weak interaction limits (unnatural and natural).

1. For strong interactions, $\frac{1}{a_0} \to 0$ in which case

$$C_0(\Lambda_c) = -\frac{2\pi^2}{m\Lambda_c} < 0 , \tag{30}$$

which is always attractive to give a zero-energy bound state.

2. For weak interactions, we can choose $|\Lambda_c| a_0 \ll 1$ for a large cutoff range and expand

$$C_0(\Lambda_c) = \frac{4\pi a_0}{m} \frac{1}{1 - \frac{2}{\pi}\Lambda_c a} \approx \frac{4\pi a_0}{m}(1 + \frac{2}{\pi} a_0 \Lambda_c) \,, \tag{31}$$

   which is what we found before doing perturbative matching.

In the weak interaction case, each successive term in the LS equation is suppressed by an additional power of $a_0\Lambda_c$. But in the strong interaction case, each vertex C_0 contributes a $1/\Lambda_c$ factor while each $I_0(k, \Lambda_c)$ contributes $(ik + \Lambda_c) \sim \Lambda_c$. The result is that each term in the expansion is equally important, which is our signal that we need to sum them.

Some further comments:

- Once again, the result that the potential $C_0(\Lambda_c)$ depends on the choice of Λ_c means that it is not unique and hence can not be measured directly in an experiment.
- We say that the potential is both *scale* and *scheme* dependent. The scale dependence is from changing Λ_c, while the scheme dependence is from choosing a sharp cutoff as a regulator (cf. choosing $C_0 e^{-(k^2+k'^2)^n/\Lambda_c^{2n}}$ with the intermediate integrations running over the full range from 0 to ∞).
- We need to use a consistent scheme for currents and many-body forces.
- In coordinate space, the potential with a finite cutoff Λ_c is a smeared out delta function, fine-tuned in the unitary limit to have a bound-state at $E = 0$.

c. More on errors in pionless EFT

There are two sources of errors in the leading-order pionless EFT:

1. From omitted terms, such as C_2, C_2', which scale as

$$\left(\frac{Q}{\Lambda_{\text{breakdown}}}\right)^2 \equiv \left(\frac{Q}{\Lambda_b}\right)^2 \sim \left(\frac{Q}{m_\pi}\right)^2 \tag{32}$$

2. From regularization artifacts, namely the finite cutoff leads to a Q^2/Λ_c^2 error, which effectively induces an effective range r_0 of order $1/\Lambda_c$. (That is, a contribution of this form corresponds to an effective range term in the ERE.)

Thus the total error should scale as the maximum of these errors:

$$\text{error} \ \sim \max\left(\left(\frac{Q}{\Lambda_b}\right)^2, \left(\frac{Q}{\Lambda_c}\right)^2, \right) \,. \tag{33}$$

This implies that as long as $\Lambda_c \geq \Lambda_b$, the regularization does not lead to errors larger than those already present from the EFT truncation.

The systematics of the errors in phase shifts can be manifested by making error plots (of the type used in ordinary numerical analysis) that plot the logarithm of the (relative) error versus the logarithm of the momentum scale. These are generally called "Lepage plots" after their introduction in Ref. [10]. (Examples for perturbative matching are in Fig. 2.) For $\Lambda_c \geq \Lambda_b$, we expect the low momentum region (i.e., well below the breakdown scale Λ_b) to be dominated by the leading power of k/Λ_b, so we get straight lines with increasing slope as we go from LO to NLO to N^{2}LO. The asymptotic intercept should roughly determine Λ_b. If we consider a *fixed* order but vary $\Lambda_c < \Lambda_b$, we expect lines with the same (low-momentum) slope, but for lower Λ_c they will have the slope inherited from the regulator. This will improve (the line moves down) until it stops improving for $\Lambda_c \approx \Lambda_b$. The non-perturbative solution is increasingly non-linear as Λ_c increases further, which could lead to a deterioration in the error. So nothing is gained in the error by taking Λ_c toward ∞ and much can be lost by making the numerical solution more difficult. (You add increasingly incorrect physics in the sum over intermediate states, the loops, which has to be canceled by the interaction. This requires larger basis sizes and increased fine tuning, both of which are usually negatives from a computational point of view unless there are other compensations.)

d. Renormalization group equation

The running of $C_0(\Lambda_c)$ with the cutoff can be expressed in the form of a differential equation, which is called a renormalization group (RG) equation. The key is to demand that the on-shell T-matrix, which is measurable, must be independent of Λ_c (for small k, because leading order):

$$\begin{aligned}
\frac{dT}{d\Lambda_c} = \frac{d}{d\Lambda_c}\frac{1}{\frac{1}{C_0(\Lambda_c)} - I_0(k,\Lambda_c)} = 0 &\iff \frac{d}{d\Lambda_c}\frac{1}{C_0} = \frac{d}{d\Lambda_c} I_0(k,\Lambda_c) \\
&\iff -\frac{1}{C_0(\Lambda_c)^2}\frac{dC_0(\Lambda_c)}{d\Lambda_c} = -\frac{m}{2\pi^2}\left(1 + \mathcal{O}(k^2/\Lambda_c^2)\right) \\
&\iff \frac{dC_0(\Lambda_c)}{d\Lambda_c} = \frac{m}{2\pi^2}\left(C_0(\Lambda_c)\right)^2 \ . \qquad (34)
\end{aligned}$$

We've neglected the k^2/Λ_c^2 correction at low k. Can you solve this equation? What is the initial condition?

How does this compare to the QCD running coupling $\alpha_s(Q^2)$? This is usually written as a function of Q^2, while the C_0 running is with respect to Λ_c; how do you account for this? Can you explain how the *source* of the running is the same? Can you devise an analogous quantity to $\Lambda_{\rm QCD}$?

e. Generalized potential

What is the most general pionless potential? In Section 6 we considered a spin-dependent term in the low-energy Lagrangian without isospin (that is, just spin-up and spin-down fermions, such as neutrons) and stated (without proof) that the term was redundant. Let's revisit such a term here

but at the level of the potential. So consider

$$V = C_S + C_T \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \ . \tag{35}$$

We will account for the antisymmetrization necessary in matrix elements of V by adding operators that exchange the coordinates:

$$V_{\text{antisym.}} = (1 - P_{12})V \tag{36}$$

where the exchange operator P_{12} acts on the relative momenta and the spin:

$$P_{12} = P_{\mathbf{k}\leftrightarrow\mathbf{k}'} P_{\text{spin}} \qquad \text{where} \qquad P_{\text{spin}} = \frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2} \ . \tag{37}$$

(Do you agree that the action of P_{12} on the momenta is to simply exchange the relative momenta?) At leading order there is no momentum dependence, so $P_{\mathbf{k}\leftrightarrow\mathbf{k}'}$ is just the identity operator, leaving

$$\begin{aligned} V_{\text{antisym.}} &= (1 - P_{\text{spin}})(C_S + C_T \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \\ &= \frac{1}{2}\left(C_S - 3C_T + (3C_T - C_S)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\right) \\ &= \begin{cases} 0 & S = 1 \\ 2(C_S - 3C_T) & S = 0 \end{cases} \end{aligned} \tag{38}$$

where we have used

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2 = 3 - 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \ . \tag{39}$$

Thus we come to the same conclusion as in Section 6, which is that there is only one linearly independent combination of C_S and C_T, namely $C_S - 3C_T$. We could, for example, choose $C_S = C_0$, $C_T = 0$ *or* $C_S = 0$, $C_T = -C_0/3$ and we would get exactly the same results.

[Question: What is $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ in spin-singlet and spin-triplet states?] [Question: How would you build construction operators that project onto spin-singlet and spin-triplet parts of a wave function?]

Next consider LO pionless EFT but with spin *and* isospin. There are now four possible operators to consider:

$$\mathbb{1},\ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,\ \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2,\ \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \ , \tag{40}$$

but only two different S-waves (singlet and triplet). As you have probably guessed, only two of these are independent and we can pick any combination. The conventional choice is:

$$V_{NN}^{\text{LO}} = C_S + C_T \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \ . \tag{41}$$

At NLO there are 14 possible operators but only 7 are linearly independent. The conventional choice is

$$\begin{aligned} V_{NN}^{\text{NLO}} =& C_2 \frac{1}{2}(k^2 + k'^2) + C_2' \mathbf{k} \cdot \mathbf{k}' + C_2^S \frac{1}{2}(k^2 + k'^2)\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_2'^S \mathbf{k} \cdot \mathbf{k}' \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \\ &+ i C_2^{\text{LS}} (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot (\mathbf{k} \times \mathbf{k}') + C_2^T \boldsymbol{\sigma}_1 \cdot (\mathbf{k}' - \mathbf{k}) \boldsymbol{\sigma}_2 \cdot (\mathbf{k}' - \mathbf{k}) + C_2'^T \boldsymbol{\sigma}_1 \cdot (\mathbf{k}' + \mathbf{k}) \boldsymbol{\sigma}_2 \cdot (\mathbf{k}' + \mathbf{k}) \ . \end{aligned} \tag{42}$$

In the second line, the first term is a spin-orbit interaction and the second and third terms lead to tensor interactions. The individual terms are explicitly non-local, because they do not depend only on the momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. However, it turns out that one can pick a different linear combination that are (almost) all functions of $\mathbf{q}$ only. This turns out to be advantageous for some applications.

f. References

[1] P. F. Bedaque, H. W. Hammer, and U. van Kolck. Effective theory for neutron-deuteron scattering: Energy dependence. *Phys. Rev. C*, 58:641–644, 1998.

[2] P. F. Bedaque, H. W. Hammer, and U. van Kolck. The three-boson system with short-range interactions. *Nucl. Phys. A*, 646:444–466, 1999.

[3] J.-W. Chen, G. Rupak, and M. J. Savage. Nucleon-nucleon effective field theory without pions. *Nucl. Phys. A*, 653:386–412, 1999.

[4] R. Furnstahl, J. V. Steele, and N. Tirfessa. Perturbative effective field theory at finite density. *Nucl. Phys. A*, 671:396–415, 2000.

[5] R. J. Furnstahl, H. W. Hammer, and N. Tirfessa. Field redefinitions at finite density. *Nucl. Phys. A*, 689:846–868, 2001.

[6] H. Georgi. On-shell effective field theory. *Nucl. Phys. B*, 361:339–350, 1991.

[7] H. W. Hammer and R. J. Furnstahl. Effective field theory for dilute fermi systems. *Nucl. Phys. A*, 678:277–294, 2000.

[8] D. B. Kaplan, M. J. Savage, and M. B. Wise. Two-nucleon systems from effective field theory. *Nucl. Phys. B*, 534:329–355, 1998.

[9] D. B. Kaplan and J. V. Steele. The Long and short of nuclear effective field theory expansions. *Phys. Rev. C*, 60:064002, 1999.

[10] G. Lepage. How to Renormalize the Schrödinger Equation. 1997.

[11] J. V. Steele and R. Furnstahl. Regularization methods for nucleon-nucleon effective field theory. *Nucl. Phys. A*, 637:46–62, 1998.

[12] U. van Kolck. Effective field theory of short range forces. *Nucl. Phys. A*, 645:273–302, 1999.