Pionless natural EFT at finite density (uniform system)

Start again with pionless scattering calculated with EFT:

\[
\left( \frac{d^2}{dt^2} + \frac{\nu^2}{2m} \right) \phi = - \frac{G_0}{3} (2\phi)^3 + \frac{C_1}{16} (2\phi^4) + \text{c.c.} + \frac{C_2}{8} (\phi^2)^{1/2} \cdot (\phi^2)
\]

\[
\phi(x) = 1 \cdot \phi_1 + \phi_2 + \phi_3 + \ldots
\]

\[
\left< \phi_1 \left< \phi_2 \right| \right> = -i C_0 \cdot -i C_2 \cdot \frac{\nu^2}{2m} \phi_1 \phi_2
\]

So now apply at $T=0$ to a system of $N$ Fermions in a box of volume $V = L^3$ with spin-isospin degeneracy $\nu$. Sum over the Fermi sea to find the non-interacting density and energy density (assuming periodic boundary conditions: $e^{iKx} = e^{-iKx}$).

\[
N = \int \frac{d^3p}{(2\pi)^3} \left( 1 - \frac{e^{iKx}}{\nu} - \frac{e^{-iKx}}{\nu} \right) = \frac{V}{\nu} \int \frac{d^3k}{(2\pi)^3} = \frac{V}{\nu} \frac{4}{\pi^2} \frac{k^3}{a^3} = \frac{4}{3} \frac{V}{\nu} \frac{k}{a}
\]

\[
e_f = \left( \frac{4}{3} \frac{V}{\nu} \frac{k}{a} \right) V^3
\]

Find the interacting energy by summing over $N_f$ Fermi sea

\[
\epsilon_{f} = \text{trace of spin-isospin on fermion lines}
\]

\[
\Rightarrow \text{one sum over } \nu \text{ with minus sign}
\]

\[
\text{two independent spin sums over } \nu \Rightarrow \alpha \nu
\]
Aside: Standard Hartree-Fock discussion for local $V(x, y) \rightarrow \psi(x, y)$

Best Slater determinant in a variational sense. 

$$\Psi_{\text{HF}} = \det \{ \phi_i(x) \}, i = 1, \ldots, N \quad \text{subject to} \quad \phi_i(x) \rightarrow \text{anti-symmetrized product wave function}$$

$$\langle \Psi_{\text{HF}} | H_{\text{HF}} | \Psi_{\text{HF}} \rangle = \sum_{i = 1}^N \left( A_{\text{i}} \sum_{j = 1}^N \nabla \phi_i^* \nabla \phi_j \right) + \frac{1}{2} \sum_{j = 1}^N \left( \sum_{k = 1}^N \phi_k^* \phi_j \right)^2 - \frac{1}{2} \sum_{j = 1}^N \left( \phi_j \phi_j^* \phi_j \right)$$

Direct (Hartree) + Exchange (Fock) diagrams in coordinate space.

Determine $\phi_i$ by varying with fixed normalization.

$$\frac{1}{\sum_{j = 1}^N \phi_j^2} \left( \langle \Psi_{\text{HF}} | H_{\text{HF}} | \Psi_{\text{HF}} \rangle - \sum_{j = 1}^N \epsilon_j \sum_{k = 1}^N \phi_k^2 \phi_j^2 \right) = 0$$

Standard Hartree-Fock equation. Fill the lowest $A$ orbitals (shell)...

If $V(x, y) = C_0 \delta(x-y)$, and $\phi(x) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-x^2}$, we recover our alternative results, [exercise].

More generally (still with local): $\{ -i \nabla + \sum_{j = 1}^N \phi_j^* (x) \} \phi(x) + \int dy V(x, y) \psi(x, y)$.

Zero range

$$\rho_{\text{HF}}(x) = \int dy V(x, y) \left( \sum_{j = 1}^N \phi_j^2 (y) \right) = \int dy V(x, y) \psi^2 (y)$$

Solve self-consistently. Potentials depend on $\phi(x)$.

$$\rho_{\text{HF}}(x, y) = -V(x,y) \left( \sum_{j = 1}^N \phi_j^* (y) \phi_j (x) \right) = -V(x,y) \rho_{\text{HF}} (x,y)$$

Density matrix.
Motivation: Most integrals in $\langle \Psi | H | \Psi \rangle$ are just normalization integrals. So focus on the ones that aren't.

Second Quantization: 1st pass

To be concrete, consider N particles in a volume $\Omega$ that we take to go at $
abla$ end, with a short-range interaction $V(x) = \phi_0(\rho(x))$.

* basis is discretized plane waves: $\chi_{\alpha}(x) = \frac{1}{\sqrt{N}} e^{i\mathbf{k}_\alpha \cdot \mathbf{x}}$ (also isospin

$\Rightarrow \eta = (\mathbf{0})$, $\eta = (0)$ and $\eta\alpha\eta = \delta_{\alpha\beta}$ spin function for spin-$\frac{1}{2}$ along $\mathbf{z}$ axis

periodic boundary conditions: $k_i = 2\pi n_i / L$, $n_i = 0, \pm 1, \pm 2, \ldots$

1st quantized $H = \sum_{i=1}^{N} \frac{\mathbf{p}^2}{2m} + \frac{1}{2} C_0 \sum_{i<j} \phi_0^2(x_i - x_j)$

2nd quantized $\hat{A} = \sum_{\alpha} \frac{\alpha^2}{2m} \hat{r}_\alpha \hat{p}_\alpha + \frac{1}{2} C_0 \sum_{i<j} \phi_0^2 \hat{r}_i \hat{r}_j$

This comes from $\langle \Psi | \hat{H} | \Psi \rangle = \frac{1}{2} \int d^3x e^{i \mathbf{k}_\alpha \cdot \mathbf{x}} \chi_{\alpha}^\dagger \chi_{\alpha}$ (Check by Fock number operator $\hat{N}$)

$= \frac{k^2}{2m} \int d^3x e^{i \mathbf{k}_\alpha \cdot \mathbf{x}} = \hat{N}$

Unsymmetrized potential energy (so $\frac{1}{2} \sum V_{\alpha\beta}$ vs. $\frac{1}{2} \sum V_{\alpha\beta}$)

$\langle R^2_{\alpha \beta}, R^2_{\alpha \beta} \rangle = \int d^3x_1 d^3x_2 e^{i \mathbf{k}_\alpha \cdot \mathbf{x}_1} \chi_{\alpha}^\dagger (\mathbf{x}_1) e^{i \mathbf{k}_\beta \cdot \mathbf{x}_2} \chi_{\beta} (\mathbf{x}_2)$

$= \frac{C_0}{N} \delta_{\alpha \beta} \delta_{\alpha \beta}$
Aside: Reference States  

Now suppose as a reference state we have the free Fermi gas $|F\rangle$.

- Call the spin-isospin degeneracy $N$. (Here $N_\alpha^L=2$.)

- All lowest momentum states up to $k_F$.

- You can check that $N = \langle F | \sum \alpha \sum \alpha \bar{c}_{\alpha \alpha} \bar{c}_{\alpha \alpha} | F \rangle = \sum_{\alpha} \Theta(k_F - k_{\alpha})$

  vacuum: number operator: $\sum \frac{\delta k_{\alpha}^3}{6\pi^2}$

- As defined so far, $\bar{c}_{\alpha \alpha} | F \rangle = 0$ for all $\alpha$

  but $\bar{c}_{\alpha \alpha} | F \rangle \neq 0$ if $k_{\alpha} < k_F$

- So with respect to reference state $|F\rangle$, we can redefine $c_{\alpha \alpha}$

  $\Rightarrow \bar{c}_{\alpha \alpha} = \Theta(k_{\alpha} - k_F) a_{\alpha \alpha} + \Theta(k_F - k_{\alpha}) b^+_{\alpha \alpha}$

  [If this is a canonical transformation, preserves commutation relations]

  But my Hamiltonian will not be normal-ordered wrt $F$ now.

  For example, if $\bar{c}_{\alpha \alpha} \bar{c}_{\beta \beta}$ are all in the Fermi sea $|k_{\alpha} < k_F \rangle$

  The 2-body potential will have $b_{\alpha \beta} b^{+\beta \alpha}$, where $\bar{c}_{\alpha \alpha} = a_{\alpha \alpha} + \Theta(k_{\alpha} - k_F) b^+_{\alpha \alpha}$

  we want to normal order wrt $|F\rangle$ as for Wick’s theorem

  $\Rightarrow$ move $b_{\alpha \beta}$ to the right.

  but $\{ b_{ij} b^{+ij} \} = 8 \delta_{ij} \Rightarrow b_{ij} b^{+ij} = b_{ij} b^{+ij}$

  $\Rightarrow b_{\alpha \beta} b^{+\alpha \beta} = \delta_{\alpha \beta} b_{\beta \beta} b^{+\beta \beta} - b_{\alpha \beta} b^{+\alpha \beta}$

  or apply Wick’s theorem

  $\Rightarrow$ sum of all contractions

  ... sum of all contractions

  ... sum of all contractions

  ... sum of all contractions

  So reference state changes where many-body contributions enter

  For 3-body, reshuffles (most important) contributions to 0-body, 1-body, 2-body,

  with much smaller 3-body, so truncation to two-body force still captures much (most) of 3-body
For plane-wave basis, the quantized form of the Hamiltonian is (from E=mc^2):

\[ H = \sum_{\mathbf{k}} \frac{\hbar^2}{2m} \mathbf{\hat{a}}_\mathbf{g}^+ \mathbf{\hat{a}}_\mathbf{g} + \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} \left( \mathbf{\hat{a}}_\mathbf{g}^+ \mathbf{\hat{a}}_\mathbf{g}^* \mathbf{\hat{a}}_\mathbf{g} \mathbf{\hat{a}}_\mathbf{g}^* \right) + \sum_{\mathbf{k},\mathbf{k}'} \left( \mathbf{\hat{a}}_\mathbf{g}^+ \mathbf{\hat{a}}_\mathbf{g}^* \mathbf{\hat{a}}_\mathbf{g} \mathbf{\hat{a}}_\mathbf{g}^* \right) \]

\[ \mathbf{\hat{a}}_{\mathbf{k}x} \mathbf{\hat{a}}_{\mathbf{k}y} \mathbf{\hat{a}}_{\mathbf{k}z} \mathbf{\hat{a}}_{\mathbf{k}+\mathbf{q}} \]

The energy is given by:

\[ E^{(q)} = \langle F|H|F \rangle = \text{direct} + \text{exchange} \]

We can use contractions or switch to normal-ordered notation (exercise: check that you get the same result) [for the interacting Fermi sea].

**Direct:**

\[ \langle F|a_{\mathbf{k}x}^+ a_{\mathbf{k}y}^+ a_{\mathbf{k}z}^+ a_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}x} a_{\mathbf{k}y} a_{\mathbf{k}z} a_{\mathbf{k}+\mathbf{q}} |F \rangle \]

\[ = \Theta(k + q_k) \delta_{k_x k_x'} \delta_{k_y k_y'} \delta_{k_z k_z'} \delta_{k+q_k k+q_k} \]

\[ \mathbf{\hat{S}}_{\mathbf{k}x} \mathbf{\hat{S}}_{\mathbf{k}y} \mathbf{\hat{S}}_{\mathbf{k}z} \mathbf{\hat{S}}_{\mathbf{k}x+\mathbf{q}} \]

\[ = \frac{1}{2} \Theta(k + q_k) \delta_{k_x k_x'} \delta_{k_y k_y'} \delta_{k_z k_z'} \delta_{k+q_k k+q_k} \]

\[ \mathbf{\hat{S}}_{\mathbf{k}x} \mathbf{\hat{S}}_{\mathbf{k}y} \mathbf{\hat{S}}_{\mathbf{k}z} \mathbf{\hat{S}}_{\mathbf{k}x+\mathbf{q}} \]

\[ \text{so no restriction from this term} \]

**Exchange:**

\[ \langle F| a_{\mathbf{k}x}^+ a_{\mathbf{k}y}^+ a_{\mathbf{k}z}^+ a_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}x} a_{\mathbf{k}y} a_{\mathbf{k}z} a_{\mathbf{k}+\mathbf{q}} |F \rangle \]

\[ = -\delta_{\mathbf{k}x \mathbf{k}z} \]

\[ \mathbf{\hat{S}}_{\mathbf{k}x} \mathbf{\hat{S}}_{\mathbf{k}y} \mathbf{\hat{S}}_{\mathbf{k}z} \mathbf{\hat{S}}_{\mathbf{k}x+\mathbf{q}} \]

\[ = -\delta_{\mathbf{k}x \mathbf{k}z} \]

\[ \frac{1}{2} \Theta(k + q_k) \delta_{k_x k_x'} \delta_{k_y k_y'} \delta_{k_z k_z'} \delta_{k+q_k k+q_k} \]

\[ \mathbf{\hat{S}}_{\mathbf{k}x} \mathbf{\hat{S}}_{\mathbf{k}y} \mathbf{\hat{S}}_{\mathbf{k}z} \mathbf{\hat{S}}_{\mathbf{k}x+\mathbf{q}} \]

\[ \text{extra minus sign from anti-commutator} \]

\[ \lim_{\mathbf{q} \to 0} \frac{1}{2} \Theta(k + q_k) \delta_{k_x k_x'} \delta_{k_y k_y'} \delta_{k_z k_z'} \delta_{k+q_k k+q_k} \]

\[ \mathbf{\hat{S}}_{\mathbf{k}x} \mathbf{\hat{S}}_{\mathbf{k}y} \mathbf{\hat{S}}_{\mathbf{k}z} \mathbf{\hat{S}}_{\mathbf{k}x+\mathbf{q}} \]

One has:

\[ \sum_{\alpha} \mathbf{S}_{\alpha x} \cdot \mathbf{S}_{\alpha x} = \sum_{\alpha} \mathbf{S}_{\alpha x} \cdot \mathbf{S}_{\alpha x} = \mathbf{S}^2 \]

\[ \left( - \frac{1}{2} \frac{1}{\nu} \mathbf{c} \mathbf{p}^2 \right) \mathbf{\hat{N}} \Rightarrow \mathbf{\hat{N}}_0 = \mathbf{\hat{S}}^{(0)} + \mathbf{\hat{E}}^{(0)} = \frac{\mathbf{C}_0 \mathbf{p}^2}{2} \left( \eta - \frac{1}{\nu} \right) \]

\[ \text{Why does this vanish when } \nu \text{?} \]
For higher order, much easier to use the Feynman rules, here for Feynman diagrams in momentum space. We'll do this in a moment; first anticipate the divergence in the next order.

Linear divergence → \[ \sigma \]

The rules will restrict the intermediate state momenta in \( \Delta \) to be above \( k_f \) while in free space there is no limit.

But the Pauli blocking is at low \( k_f \) : 0 to \( k_f \) is excluded (IR physics) while the sensitivity to UV is near \( \Lambda_c \) (which we assume is greater than \( k_f \)).

We can see this explicitly by writing the divergent part

\[ \frac{\hbar c}{k_f} \sum_{\text{sp}} \frac{1}{q^2} = \int \frac{d^3 k_f}{(2\pi)^3} \frac{1}{q^2} - \int \frac{d^3 k_f}{(2\pi)^3} \frac{1}{q^2} \]

but this is what \( e^0(\Lambda_c) \) cancels

\[ \Rightarrow \quad \text{has no linear divergence!} \]

\( e^0(\Lambda_c) \) cancels the \( \Lambda_c \) dependence in \( e^1 \) cancels the \( \Lambda_c \) dependence in \( \sigma \)

Moral: Finite density is IR physics, so cut-off terms for UV physics work as in free space to remove sensitivity to cut-off.

\[ \Rightarrow \quad \text{we can do IR UV renormalization in free space} \]

and then finite density is automatically renormalized.
 Feynman rules for energy density at $T=0$: n-th order of $E-E_0$.

- Draw all distinct, fully connected diagrams with $n$ vertices.
- Each line is assigned a non-relativistic, four-momentum $k = (k_0, k)$.
- Four-momentum is conserved at each vertex. Internal frequency lines get a factor $e(k_0 - k)/a_{k_0}$.

$$\gamma = \left( \frac{G(k_0 - k)}{k_0 - k} \right)^n$$

- The vertex lines have spin (and isospin more generally) indices.

$$\to \begin{array}{c}
\times \\
\sigma
\end{array} 
\Rightarrow (\bar{\sigma} \otimes \sigma + \Sigma \otimes \sigma)$$

- Reset minus sign for exchange elsewhere.

- Do spin summations and $\sum_{\sigma} \to -1$ for each closed fermion loop.

- Integrate $\frac{dk_0}{\sqrt{k_0}}$ over independent momenta $(d\vec{k}_i = d\vec{k}_0 d\vec{k}_i)$.

- For $\bar{\sigma}_k$, assign $e^{i\theta_{\bar{\sigma}}}$ ("feynman"").

- Frequency integral -> back to time-independent results, but all time orderings.

- Multiply by a symmetry factor $i/S! \left( \frac{1}{n!} \right)$, where $S$ is the number of vertex permutations and $n$ is the number of equivalent $\bar{\sigma}$-tuples of lines (lines that begin and end at the same vertices with the same direction and arrow).

- Anomalous diagrams, with $G(\vec{k}_f - \vec{k}_0) G(\vec{k}_0 - \vec{k}_f) = 0$.

- Power counting rules when using dimensional regularization with $\overline{\text{MS}}$.

1. For every propagator line: $k_f^2 = k_f^2 = k_f^2 = k_f^2$.
2. For every loop integration: $(k_f/m) k_f^2 = k_f^2/m$.
3. For every $n$-body vertex with $2n$ derivatives: $k_f^2/m^{4+2n}$.

---

Note: The above text is a hand-written note on Feynman rules for energy density at $T=0$, with references to the integration of momenta and symmetry factors. The note also includes power counting rules for $n$-body vertices with $2n$ derivatives.
A diagram with $V_{3i}$ n-body vertices scales as $k_F^p$ where

$$p = 5 + \sum_{n=2}^{\infty} \sum_{\ell=0}^{(3n+2\ell-5)} V_{3i}^n \sim n\text{-body vertex}$$

(same as in free space with $E = \# \text{ of external lines} = 0$.)

- $\bigcirc \bigotimes \Rightarrow V_0^2 = 1 \Rightarrow p = 5 + (3 \cdot 2 + 2 \cdot 0 - 5) \cdot 2 = 6 \Rightarrow \Theta(k_F^6)$
- $\bigcirc \Rightarrow V_0^2 = 2 \Rightarrow p = 5 + (3 \cdot 2 + 2 \cdot 0 - 5) \cdot 2 = 7 \Rightarrow \Theta(k_F^7)$

Exercise questions:

- Why does the formula for $\beta$ ensure that $k_F^p$ has always at least $p=6$?
- Why does switching a vertex for one with more derivatives always increase $\beta$?
- Why does increasing the number of internal lines increase $\beta$?
- See examples of diagrams and check claimed $\beta$.

5) a perturbative expansion in $\frac{k_F}{\Lambda_0} < \Lambda_0$, where $a_0, \Lambda_0 \sim \frac{1}{\Lambda_0}$, breakdown scale.

- Does this formalism apply to nuclear systems?
- Low density: scattering length physics critical; natural or unnatural?
- Higher density: what about pion? Friday looks like Skyrme EDX?
Take-away points about naturalness of EFT at finite density

- Low resolution view - coarse-grained impressions

1) Renormalization & UV in free space carries over to finite density

\[ \text{high-momentum physics} \quad \Rightarrow \quad \text{low-momentum physics} \]

\[ \Rightarrow \text{no new divergences} \Rightarrow \text{no new sensitivities to } \Lambda_c \]

2) Energy (density) can be calculated from Feynman diagrams

- Feynman rules \( \Rightarrow \) analogous to those from relativistic field theory
- Integrate over both frequency \( k_0 \) and 3-momentum \( \vec{k} \)
- Propagator from \( 1^+ (i \frac{\partial}{\partial t} + \vec{p}^2 / 2m)^2 \)

\[ C^0 = \text{inverse of } \]

by going to eigenbasis

\[ \Rightarrow \text{take } 2 \alpha e^{-i k x} \Rightarrow (i \frac{\partial}{\partial t} + \frac{\vec{p}^2}{2m})^2 \Rightarrow k_0 - \frac{\vec{p}^2}{2m} = k_0 - \omega_k \]

Inverse boundary condition

\[ \theta(k_0 - \omega_k) \text{ for particle; propagate forward in time} \]

\[ \theta(k_0 + \omega_k) \text{ for hole; propagate backward in time} \]

\( \Rightarrow \) propagator

3) Integration over frequency \( \Rightarrow \) pick up poles

\[ L^0 \text{ to } \sum \frac{e^{i k x}}{i \epsilon + \epsilon} \]

\[ = \sum \frac{1}{i \epsilon + \epsilon} \theta(k_0 - \omega_k) \Rightarrow \text{residue at } \epsilon = \theta(k_0 - \omega_k) \]

\[ \Rightarrow \text{residue at } \epsilon = \theta(k_0 - \omega_k) \text{ in upper half plane} \]
Power counting example \implies systematic finite example.

- Diagrams scale as \( \left( \frac{k^4}{\Lambda^4} \right)^\beta \) with \( \beta = 5 + \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (3n+2i-5) \cdot V_{0+i} \).

- \((3n+2i-5) \cdot V_{0+i} \implies \beta = 6 \)

- Switch vertex for one with more dynamics.

- Add a similar vertex.

- 3-body? \( \propto k^6 \propto g^3 \) vs. \( \propto j^3 = k^3 \).

\( n=3, i=0, V^2 = 1 \implies \beta = 5 + 3 + 0 = 9 \)

\( \implies \) a finite \# of diagrams contribute at each order.

- Power series? \( \text{No, because term } \propto (1+2)(k^4\Lambda^4) \text{ is } \text{non-analytic} \).

- 3-body needed!

- An academic exercise?

- Is this like low-density reionization matter?

- No: if \( k_0^2 \gg \Lambda^4 \), then must sum all diagrams with \( k_0 \).

- Only numerically (at present).

- Anything like higher density? Don't we resolve pairs? \( \text{Yes} \).