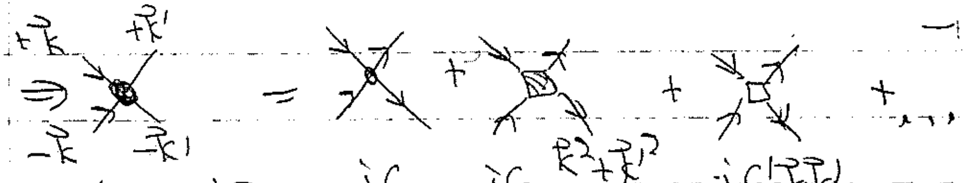
Pionless natural EFT at finite density (uniform system)

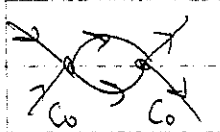
- Start again with pionless scattering calculated with EFT:

$$\mathcal{L}_{\text{eff}} = \psi^\dagger\left[i\frac{\partial}{\partial t} + \frac{\vec\nabla^2}{2m}\right]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 + \frac{C_2}{16}\left[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{h.c.}\right] + \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi)$$
$$-\frac{D_0}{6}(\psi^\dagger\psi)^3$$

total $\vec P = 0$



$-i\langle \vec k'|V_{\text{EFT}}|\vec k\rangle \quad -iC_0 \quad -iC_2\frac{k^2+k'^2}{2} \quad -iC_2'\,\vec k\cdot\vec k'$



$$\Rightarrow \int^{\Lambda_c}\frac{d^3q}{(2\pi)^3}\frac{1}{k^2-q^2+i\varepsilon} \longrightarrow \frac{\Lambda_c}{2\pi^2} - \frac{ik}{4\pi} + O\!\left(\frac{k^2}{\Lambda_c}\right) \quad \text{with cutoff } \Lambda_c$$

$\boxed{\text{changes in } \Lambda_c \text{ absorbed into } C_0^{(1)}(\Lambda_c)}$

or $$\Rightarrow \int\frac{d^Dq}{(2\pi)^3}\frac{1}{k^2-q^2+i\varepsilon} \xrightarrow{D\to 3} -\frac{ik}{4\pi} \quad \text{with dimensional regularization and MS}$$

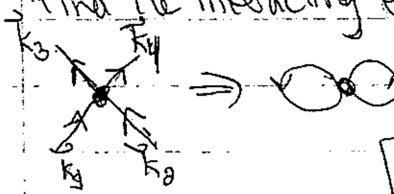
- So now apply at $T=0$ to a system of N fermions in a box of volume $\Omega = L^3$ with spin-isospin degeneracy ν.
- Sum over the Fermi sea to find the non-interacting density and energy density (assuming periodic boundary conditions: $e^{ikx} = e^{ik(x+L)}$ $\Rightarrow k_nL = 2\pi n$)

$\boxed{\text{rule: } \sum_{\vec k} \to \frac{\Omega}{(2\pi)^3}\int d^3k}$

$$\rho = \frac{N}{\Omega} = \frac{\nu}{\Omega}\sum_{\vec k}^{k_F} 1 = \frac{\nu}{\cancel{\Omega}}\frac{\cancel{\Omega}}{(2\pi)^3}\int^{k_F} d^3k = \frac{\nu k_F^3}{6\pi^2} \;;\quad \frac{E^{(0)}}{\Omega} = \frac{\nu}{\Omega}\sum_{\vec k}^{k_F}\frac{k^2}{2m} = \frac{3}{5}\frac{k_F^2}{2m}\rho \quad \text{[exercise]}$$

$\Rightarrow k_F = \left(\frac{6\pi^2}{\nu}\rho\right)^{1/3}$ $\qquad \frac{k_F^2}{2m}$: ϵ_F Fermi energy

- Find the interacting energy by summing over the Fermi sea

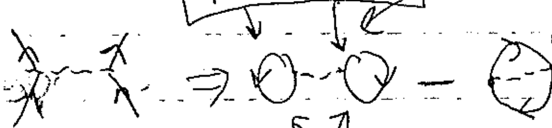


$$\mathcal{E}_{\text{LO}} = \frac{C_0}{2}\nu(\nu-1)\left(\frac{1}{\Omega}\sum_{\vec k}^{k_F} 1\right)^2 = \frac{C_0}{2}\left(1-\frac{1}{\nu}\right)\rho^2$$

ν: Hartree, -1: Fock; $\boxed{\text{subtract out overcount}}$; trace of spin-isospin on fermion lines

$\boxed{\nu = 2:\ \text{spin} \uparrow \text{or} \downarrow \times \text{spin} \uparrow \text{or} \downarrow}$

If we resolve ●:



two independent spin sums are $\nu \Rightarrow \nu^2$

← one sum over ν with minus sign $\Rightarrow \propto \nu$

Aside: Standard Hartree-Fock discussion for local $V(\vec{x},\vec{y}) \to V(\vec{x}-\vec{y})$

$\vec{x}_i \vec{y}$ ... not relative coordinates here

...ery ...ited ... compared ... CI or ... or ... methods:

- Best Slater determinant in a variational sense

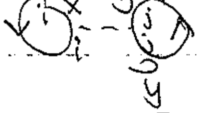
$$|\Psi_{HF}\rangle = \det\{\phi_i(\vec{x}),\ i=1\cdots A\} \qquad \vec{x}\equiv\{\vec{r},\sigma,\tau\}$$

← antisymmetrized product wave function

$$\langle\Psi_{HF}|\hat{H}|\Psi_{HF}\rangle = \sum_{i=1}^{A}\frac{1}{2m}\int d\vec{x}\,\nabla\phi_i^*\cdot\nabla\phi_i$$

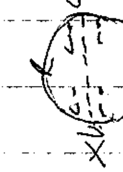
Feynman diagrams in coordinate space: mean fields

$$+\frac{1}{2}\sum_{i,j=1}^{A}\int d\vec{x}\int d\vec{y}\,|\phi_i(\vec{x})|^2\,V(\vec{x},\vec{y})\,|\phi_j(\vec{y})|^2$$


direct (Hartree)

local potential (later we'll need non-local) more integrations

$$-\frac{1}{2}\sum_{i,j=1}^{A}\int d\vec{x}\int d\vec{y}\,\phi_i^*(\vec{x})\phi_i(\vec{y})V(\vec{x},\vec{y})\phi_j^*(\vec{y})\phi_j(\vec{x})$$


exchange (Fock)

$\sum_i \phi_i^*(\vec{x})\phi_i(\vec{y}) \to$ one particle density matrix $\rho(\vec{y},\vec{x})$

- Determine the ϕ_i by varying with fixed normalization

$$\frac{\delta}{\delta\phi_i^*(\vec{x})}\Big(\langle\Psi_{HF}|\hat{H}|\Psi_{HF}\rangle - \sum_{j=1}^{A}\varepsilon_j\int d\vec{y}\,|\phi_j(\vec{y})|^2\Big) = 0$$

← Lagrange multiplier

$\Rightarrow$ standard Hartree-Fock equation. * Fill the lowest A orbitals (for closed shell)

- If $V(\vec{x},\vec{y}) = C_0\,\delta^3(\vec{x}-\vec{y})$, and $\phi_i(\vec{x}) \to \frac{1}{\sqrt{V}}e^{i\vec{k}_i\cdot\vec{x}}\eta_\alpha$,

we recover our alternative results, [exercise]

- zero range

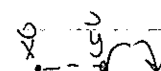
non-local potential

- More generally (still with local): $\Big\{-\frac{\hbar^2}{2m}\nabla^2 + \Gamma_H(\vec{x})\Big\}\phi_i(\vec{x}) + \int d\vec{y}\,\Gamma_F(\vec{x},\vec{y})\phi_i(\vec{y}) = \varepsilon_i\phi_i(\vec{x})$

where

solve self-consistently (potentials depend on ϕ_i's)

$$\Gamma_H(\vec{x}) = \int d\vec{y}\,V(\vec{x},\vec{y})\sum_{j=1}^{a}|\phi_j(\vec{y})|^2 = \int d\vec{y}\,V(\vec{x},\vec{y})\,\rho(\vec{y})$$



$$\Gamma_F(\vec{x},\vec{y}) \equiv -V(\vec{x},\vec{y})\sum_{j=1}^{a}\phi_j^*(\vec{y})\phi_j(\vec{x}) = -V(\vec{x},\vec{y})\,\underbrace{\rho(\vec{x},\vec{y})}_{\text{density matrix}}$$

- more complicated with non-local potential

Motivation: Most integrals in $\langle \Psi_{HF} | \hat{H} | \Psi_{HF} \rangle$ are just normalization integrals, so focus on the ones that aren't.

8-3

Second Quantization: 1^{st} pass

To be concrete, consider N neutrons in a volume Ω that we take to ∞ at the end, with a short-range interaction $V(\vec{x}) = C_0 \delta^3(\vec{x})$.

- basis is discretized normalized plane waves: $\psi_{\vec{k}\alpha}(\vec{x}) = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\cdot\vec{x}} \eta_\alpha$ — also isospin; η_α: spin function for spin-1/2 along $\hat{z}$ axis

$\Rightarrow \eta_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}, \ \eta_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$ and $\eta_\alpha^\dagger \eta_\beta = \delta_{\alpha\beta}$

$\alpha = 0,1;\ \beta = 0,1$

- periodic boundary conditions $k_i = 2\pi n_i / L$, $n_i = 0, \pm 1, \pm 2, \ldots$

- 1^{st} quantized $H = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \frac{1}{2} C_0 \sum_{i \neq j}^{N} \delta^3(\vec{x}_i - \vec{x}_j)$

one-body; double-counting pairs; no self-interaction; two-body

- 2^{nd} quantized

$$\hat{H} = \sum_{\vec{k}\alpha} \frac{k^2}{2m} a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} + \frac{1}{2}\frac{C_0}{\Omega} \sum_{\vec{k}_1\alpha_1} \sum_{\vec{k}_2\alpha_2} \sum_{\vec{k}_3\alpha_3} \sum_{\vec{k}_4\alpha_4} a^\dagger_{\vec{k}_1\alpha_1} a^\dagger_{\vec{k}_2\alpha_2} a_{\vec{k}_4\alpha_4} a_{\vec{k}_3\alpha_3} \times \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4}$$

"normal-ordered" with respect to vacuum $\Rightarrow$ all a's to right of $a^\dagger$'s $\Rightarrow$ ready to apply Wick's theorem

$\boxed{\text{add up the kinetic energy of each occupied mode, as checked by the number operator } a^\dagger_{k\alpha} a_{k\alpha}}$

This comes from

$$\langle \vec{k}_1\alpha_1 | \frac{\hat{p}^2}{2m} | \vec{k}_2\alpha_2 \rangle = \frac{1}{\Omega}\int d^3x\, e^{-i\vec{k}_1\cdot\vec{x}} \eta^\dagger_{\alpha_1} \left(\frac{-\nabla^2}{2m}\right) e^{i\vec{k}_2\cdot\vec{x}} \eta_{\alpha_2}$$

$$= \frac{k_2^2}{2m\Omega} \delta_{\alpha_1\alpha_2} \int d^3x\, e^{i(\vec{k}_2 - \vec{k}_1)\cdot\vec{x}} \leftarrow \Omega \delta_{\vec{k}_1,\vec{k}_2}$$

$$= \frac{k_2^2}{2m} \delta_{\alpha_1\alpha_2} \delta_{\vec{k}_1\vec{k}_2}$$

Unsymmetrized potential energy (so $\frac{1}{2}\sum V_{unsym}$ vs. $\frac{1}{4}\sum V_{sym}$)

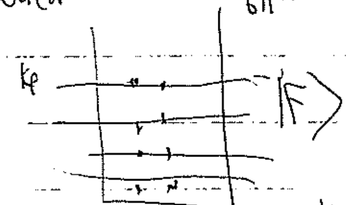
$$\langle \vec{k}_1\alpha_1, \vec{k}_2\alpha_2 | V | \vec{k}_3\alpha_3 \vec{k}_4\alpha_4 \rangle = \frac{1}{\Omega^2} \iint d^3x_1\, d^3x_2\, e^{-i\vec{k}_1\cdot\vec{x}_1} \eta^\dagger_{\alpha_1}(1) e^{-i\vec{k}_2\cdot\vec{x}_2} \eta^\dagger_{\alpha_2}(2) \times C_0 \delta^{(3)}(\vec{x}_1 - \vec{x}_2) e^{i\vec{k}_3\cdot\vec{x}_1} \eta_{\alpha_3}(1) e^{i\vec{k}_4\cdot\vec{x}_2} \eta_{\alpha_4}(2)$$

particle 1   particle 2   from WF's

$$= \frac{C_0}{\Omega} \delta_{\alpha_1\alpha_2} \delta_{\alpha_3\alpha_4} \delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4}$$

Aside: Reference states

- Now suppose as a reference state we have the free Fermi gas $|F\rangle$
  - Call the spin-isospin degeneracy ν (here $\nu = 2$)
  - Fill lowest momentum states up to k_F.
    - you can check that $N = \langle F| \sum_{\vec{k}\alpha} a^{\dagger}_{\vec{k}\alpha} a_{\vec{k}\alpha} |F\rangle = \sum_{\vec{k}\alpha} \theta(k_F - k)$ ← number operator; $\to N \frac{\nu k_F^3}{6\pi^2}$



- As defined so far, $a_{\vec{k}\alpha}|0\rangle = 0$ for all $\vec{k},\alpha$ (vacuum)
  - but $a_{\vec{k}\alpha}|F\rangle \neq 0$ if $|\vec{k}| < k_F$

with reference to $|F\rangle \Rightarrow a_{\vec{k}\alpha}$ is destruction operator for $|\vec{k}| > k_F$, but creation operator (of a hole) for $|\vec{k}| < k_F$

- So with respect to reference state $|F\rangle$, we can redefine $a_{\vec{k}\alpha}$

$$\Rightarrow a_{\vec{k}\alpha} \to \theta(|\vec{k}| - k_F)\, a_{\vec{k}\alpha} + \theta(k_F - \vec{k})\, b^{\dagger}_{-\vec{k}\alpha}$$ ← just relabeling!

[This is a canonical transformation ⇒ preserves anti commutation relations]

- But my Hamiltonian will not be normal-ordered wrt F now. For example, if $\vec{k}_1, \ldots, \vec{k}_4$ are all in the Fermi sea $|\vec{k}_i| \le k_F$, then the 2-body potential will have $b_1 b_2 b_3^{\dagger} b_4^{\dagger}$ (where $1 \equiv \vec{k}_1\alpha_1$)
  - we want to normal order wrt $|F\rangle$ as for Wick's theorem
    $\Rightarrow$ move b_1, b_2 to the right.
  - but $\{b_i, b_j^{\dagger}\} = \delta_{ij} \Rightarrow b_2 b_3^{\dagger} = \delta_{23} - b_3^{\dagger} b_2$

$$\Rightarrow b_1 b_2 b_3^{\dagger} b_4^{\dagger} = \delta_{23} b_1 b_4^{\dagger} - b_1 b_3^{\dagger} b_2 b_4^{\dagger}$$ with $b_1 b_3^{\dagger} = (\delta_{13} - b_3^{\dagger} b_1)$

$$= \underbrace{\delta_{23}\delta_{14}}_{\text{0-body}} - \underbrace{\delta_{23} b_4^{\dagger} b_1 - \delta_{24} b_1 b_3^{\dagger}}_{\text{1-body}} + b_1 b_3^{\dagger} b_4^{\dagger} b_2$$

1-body normal ordered

now bring b_1 through ⇒ more 0-body, 1-body and finally 2-body $b_3^{\dagger} b_4^{\dagger} b_1 b_2$

or, apply Wick's theorem ⇒ sum of all contractions

- So reference state changes where many-body contributions enter.
- For 3-body, reshuffles (most important) contributions to 0-body, 1-body, 2-body, with much smaller 3-body, so truncation to two-body force still captures much (most) of 3 body.

volume $\Omega \equiv V$

The 2nd quantized form for the Hamiltonian for plane-wave basis is (from (8-3))

$$\hat{H} = \sum_{\vec{k}\alpha} \frac{k^2}{2m} a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} + \frac{1}{2}\frac{C_0}{\Omega} \sum_{\vec{k}_1\alpha_1}\sum_{\vec{k}_2\alpha_2}\sum_{\vec{k}_3\alpha_3}\sum_{\vec{k}_4\alpha_4} a^\dagger_{\vec{k}_1\alpha_1} a^\dagger_{\vec{k}_2\alpha_2} a_{\vec{k}_4\alpha_4} a_{\vec{k}_3\alpha_3} \underbrace{\delta_{\alpha_1\alpha_3}}_{\text{set } \alpha_3=\alpha_1} \underbrace{\delta_{\alpha_2\alpha_4}}_{\text{set } \alpha_4=\alpha_2} \delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4}$$

$\Rightarrow E^{(1)} = \langle F|\hat{H}_1|F\rangle =$ direct + exchange

[Filled non-interacting Fermi sea]

· we can do these contractions or switch to normal-ordering wrt $|F\rangle$ [exercise; check that you get the same result]

direct: $\langle F| a^\dagger_{\vec{k}_1\alpha_1} a^\dagger_{\vec{k}_2\alpha_2} a_{\vec{k}_4\alpha_2} a_{\vec{k}_3\alpha_1} |F\rangle \, \delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4}$

$a^\dagger_{\vec{k}_2\alpha_2} a_{\vec{k}_4\alpha_2}$

$$= \Theta(k_F - |\vec{k}_2|)\,\delta_{\vec{k}_2\vec{k}_4}\,\underbrace{\delta_{\alpha_2\alpha_2}}_{\nu}\,\delta_{\vec{k}_1\vec{k}_3}\,\underbrace{\delta_{\alpha_1\alpha_1}}_{\nu} \times \Theta(k_F - |\vec{k}_1|)\,\delta_{\vec{k}_1+\vec{k}_2, \underbrace{\vec{k}_3+\vec{k}_4}_{\vec{k}_1+\vec{k}_2}}$$

so no restriction from this term

$$\Rightarrow E^{(1)}_{\text{direct}} = \frac{1}{2}\frac{C_0}{\Omega}\left(\frac{\Omega}{(2\pi)^3}\int d^3k_1\, \nu\, \Theta(k_F - |\vec{k}_1|)\right)\left(\frac{\Omega}{(2\pi)^3}\int d^3k_2\, \nu\, \Theta(k_F - |\vec{k}_2|)\right) = \left(\frac{1}{2}C_0\rho^2\right)\Omega$$

$\Omega = A$

energy density $\Rightarrow \mathcal{E}^{(1)} = \dfrac{E^{(1)}_{\text{direct}}}{\Omega} = \dfrac{1}{2}C_0\rho^2 \propto k_F^6$

$\vec{k}_1$, α_1, α_2, $\vec{k}_2$, $\vec{q}=0$

exchange $\langle F| a^\dagger_{\vec{k}_1\alpha_1} a^\dagger_{\vec{k}_2\alpha_2} a_{\vec{k}_4\alpha_2} a_{\vec{k}_3\alpha_1} |F\rangle \, \delta_{\vec{k}_1+\vec{k}_2, \vec{k}_3+\vec{k}_4}$

← 1 extra minus sign from anticommutator

$$-\Theta(k_F - |\vec{k}_1|)\,\delta_{\vec{k}_1\vec{k}_4}\,\delta_{\alpha_1\alpha_2}\,\Theta(k_F - \vec{k}_2)\,\delta_{\vec{k}_2\vec{k}_3}\,\delta_{\alpha_1\alpha_2}\,\delta_{\vec{k}_1+\vec{k}_2, \underbrace{\vec{k}_3+\vec{k}_4}_{\vec{k}_2+\vec{k}_1}}$$

independent of momentum for C_0!

$$\sum_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_2} = \sum_{\alpha_1}\delta_{\alpha_1\alpha_1} = \nu$$

one less power of ν

$$= \left(-\frac{1}{2}\frac{1}{\nu}C_0\rho^2\right)\Omega \quad\Rightarrow\quad \mathcal{E}_{LO} = \mathcal{E}^{(1)}_{\text{direct}} + \mathcal{E}^{(1)}_{\text{exchange}} = \frac{C_0}{2}\left(1 - \frac{1}{\nu}\right)\rho^2$$

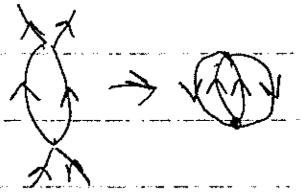
Why does this vanish when $\nu = 1$?

derived in 2nd quantization or from path integrals

For higher order, much easier to use the Feynman rules, here for Feynman diagrams in momentum space. Well do this in a moment; first anticipate the divergence in the next order.

also available for coordinate space or in a basis,

linear divergence



The rules will restrict the intermediate state momenta in to be above k_F while in free space there is no limit.

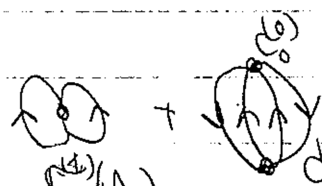
- But the Pauli blocking is at <u>low k</u>: 0 to k_F is excluded (IR physics) while the sensitivity to UV is near Λ_c. (which we assume is greater than k_F)
- We can see this explicitly by writing the divergent part as

$$\int_{k_F}^{\Lambda_c} \frac{d^3q}{(2\pi)^3} \frac{-1}{q^2} = \int_0^{\Lambda_c} \frac{d^3q}{(2\pi)^3}\left(-\frac{1}{q^2}\right) - \int_0^{k_F} \frac{d^3q}{(2\pi)^3}\left(\frac{1}{q^2}\right)$$

but this is what

$C_0^{(1)}(\Lambda_c)$ cancels

$\Rightarrow$



$C_0^{(1)}(\Lambda_c)$ + $C_0^{(0)}$ $C_0^{(0)}$ has no linear divergence!

(The Λ_c dependence in $C_0^{(1)}$ cancels the Λ_c dependence in )

Moral: Finite density is IR physics, so counterterms for UV physics work as in free space, to remove sensitivity to cutoff.

$\Rightarrow$ we can do the UV renormalization in free space and then finite density is automatically renormalized.

$\frac{\text{energy}}{\text{particle}} \; \frac{E}{N} = \frac{E}{V} \cdot \frac{V}{N} = \frac{\mathcal{E}}{\rho}$

energy/volume

energy density

8-6

s an xample of finite ρ rules, ut only for uniform system

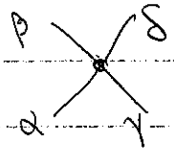
Feynman rules for energy density at $T=0$: n^{th} order of $\mathcal{E}-\mathcal{E}_0$ ← kinetic energy

- Draw all distinct fully connected diagrams with n vertices

① each line is assigned a non-relativistic four momentum $\tilde{k} \equiv (k_0, \vec{k})$ ← frequence and four-momentum is conserved at each vertex. Internal lines get the factor

$$iG_0(\tilde{k})_{\alpha\beta} = i\delta_{\alpha\beta}\left(\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon} + \frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right) \qquad \omega_k \equiv \frac{k^2}{2m}$$

② The vertex lines have spin (and isospin more generally) indices



$$\rightarrow (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$$ [if spin independent; otherwise there are σ's]

↖ get minus sign for exchange elsewhere

③ Do spin summations and $\delta_{\alpha\alpha} \rightarrow -\nu$ for each closed fermion loop

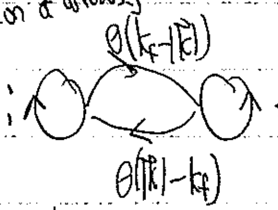
④ Integrate $\int \frac{d^4\tilde{k}}{(2\pi)^4}$ over independent momenta $(d^4\tilde{k}_i = dk_{i0}\, d^3\vec{k}_i)$

, for ⟲$_{k_0}$ assign $e^{ik_0 0^+}$ ("tadpoles")

- after frequency integral → back to time-independent results, but all time orderings,

⑤ multiply by a symmetry factor $i/S\prod_{l=2}^{l_{max}}(l!)^m$ where S is the number of vertex permutations and m is the number of equivalent l-tuples of lines, ← [lines that begin and end at the same vertices with the same direction of arrows.]

- anomalous diagrams, with $\theta(|\vec{k}|-k_F)\,\theta(k_F-|\vec{k}|)=0$; (only at $T=0$)



$= 0$

- Power counting rules when using dimensional regularization with MS

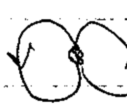
1. for every propagator line: $\frac{m}{k_F^2}$
2. for every loop integration: $(k_F^2/m)\,k_F^3 = k_F^5/m$
3. for every n-body vertex with $2i$ derivatives: $k_F^{2i}/m\Lambda^{2i+3n-5}$

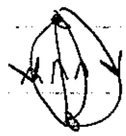
} very clean if dimensional regularization used

- A diagram with V_{2i}^{n} n-body vertices scales as k_F^{β} where

$$\beta = 5 + \sum_{n=2}^{\infty} \sum_{i=0}^{\infty} (3n + 2i - 5) V_{2i}^{n} \quad \begin{array}{l} \leftarrow n\text{-body vertex} \\ \leftarrow 2i \text{ derivatives} \end{array}$$

(same as in free space with $E = \#$ of external lines $= 0$!)

-  $\Rightarrow V_0^2 = 1 \Rightarrow \beta = 5 + (3\cdot 2 + 2\cdot 0 - 5)\cdot 1 = 6$
$\Rightarrow \mathcal{O}(k_F^6)$

-  $\Rightarrow V_0^2 = 2 \Rightarrow \beta = 5 + (3\cdot 2 + 2\cdot 0 - 5)\cdot 2 = 7 \Rightarrow \mathcal{O}(k_F^7)$

$$\mathcal{E} = \underbrace{\rho \frac{k_F^2}{2m}}_{\sim k_F^5} (\nu - 1) \frac{4}{35\pi^2} \underbrace{(11 - 2\ln 2)}_{\text{pure geometric factor}} (k_F a_0)^2$$

Exercise questions:
- Why does the formula for β ensure that k_F^{β} has always at least $\beta = 6$?
- Why does switching a vertex for one with more derivatives always increase β?
- Why does increasing the number of internal lines increase β?
- See examples of diagrams and check claimed β.

$\Rightarrow$ a perturbative expansion in $\frac{k_F}{\Lambda_b} \leftarrow$ not Λ_c! where $a_0, r_0 \sim \frac{1}{\Lambda_b}$

(Λ_b: breakdown scale)

- Does this formalism apply to nuclear systems?
  - low density: scattering length physics critical: natural or unnatural?
  - higher density: what about pions? Friday: looks like Skyrme EDF!

Take-away points about natural pionless EFT at finite density

- Low resolution view — coarse-grained impressions

(1) Renormalization of UV in free space carries over to finite density
$\underbrace{\text{Renormalization of UV}}_{\text{high-momentum physics}}$ … $\underbrace{\text{finite density}}_{\text{low-momentum physics}}$
$\Rightarrow$ no new divergences $\Rightarrow$ no new sensitivities to Λ_c

(2) Energy (density) can be calculated from Feynman diagrams without external lines

reduce a diagram to an integral $\Rightarrow$ numerical in general

- Feynman rules $\to$ analogous to those from relativistic field theory
  - integrate over both frequency k_0 and 3-momentum $\vec{k}$
  - propagator from $\psi^\dagger\left(i\frac{\partial}{\partial t}+\nabla^2/2m\right)\psi$ $\leftarrow G^0 \propto$ inverse of ( )

find inverse by going to eigenbasis $\Rightarrow$

- take $\psi \propto e^{-ik_0 t}\, e^{i\vec{k}\cdot\vec{x}} \Rightarrow \left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)\psi \to k_0-\frac{k^2}{2m} \equiv k_0-\omega_k$

inverse $\xrightarrow[\text{conditions}]{\text{boundary}}$ $\left(\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon}+\frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right)$ "Feynman propagator"

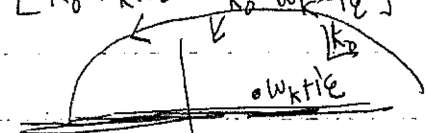
particle: propagate forward in time

hole: propagate backward in time

(cf. positive vs. negative energy Dirac propagator)

- integration over frequency $\Rightarrow$ pick up poles

LO:  $\frac{1}{2}C_0\left(1-\frac{1}{\nu}\right)\left(i\nu\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left[\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon}+\frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right]e^{ik_0 0^+}\right)^2$



close in upper half plane

$\Rightarrow$ residue $2\pi i\,\theta(k_F-|\vec{k}|)$

$= \frac{1}{2}C_0\left(1-\frac{1}{\nu}\right)C_0\rho^2$

[since $\rho = \nu\int\frac{d^3k}{(2\pi)^3}\theta(k_F-|\vec{k}|)$]

(3) Power counting example ⇒ systematic finite ρ example

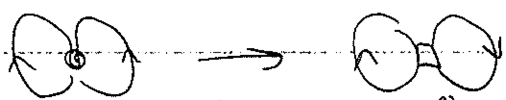
energy density

· diagrams scale as $\left(\frac{k_f}{\Lambda_b}\right)^{\beta}$ with $\beta = 5 + \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(3n+2i-5)\,V_{2i}^{n}$

n-body vertex — $2i$ derivatives — # of $n, 2i$ vertices

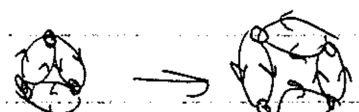
· $(\underset{n\geq 2}{3n} + \underset{2i\geq 0}{2i} - 5)\underset{\geq 1}{V_{2i}^{n}} \geq 1 \Rightarrow \beta \geq 6$

· Switch vertex for one with more derivatives



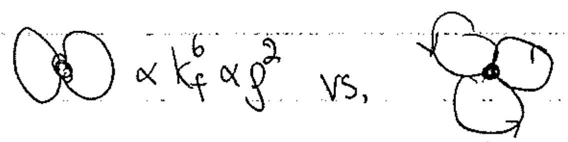
$C_0 \Rightarrow \beta = 6$ $\quad$ $C_2(k^2+k'^2) \Rightarrow 2i=2 \Rightarrow \beta = 8$ $\quad \Rightarrow 2i\uparrow, \beta\uparrow$ always

· Add a similar vertex



$V_0^2 = 3 \Rightarrow \beta = 8$ $\quad$ $V_0^2 = 4 \Rightarrow \beta = 9 \Rightarrow V_{2i}^{n}\uparrow, \beta\uparrow$ always

· 3-body?



$\propto k_f^6 \propto \rho^2$ vs. $\propto \rho^3 = k_f^9$

D_0

$n=3, i=0, V_0^3 = 1 \Rightarrow \beta = 5 + 3\cdot 3 + 2\cdot 0 - 5 = 9$ ✓

⇒ a finite # of diagram contribute at each order.

· power series? No, because term with $(\nu-2)(\nu-1)(k_f a_0)^4 \ln(k_f a_0)$ non-analytic

· 3-body needed!

· An academic exercise?

· Is this like low-density neutron matter?

· no: if $a_0 \gg r_0$, then must sum all diagrams with C_0

⇒ only numerically (at present)

· Anything like higher density? Don't we resolve pions?