Physics 8805: Nuclear Few- and Many-Body Physics
Discussion questions and exercises for Nuclear Forces 1

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1. Two minute and discussion questions.

   (a) Why is a term like $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\tau_2}$ not allowed in the nuclear potential?

   (b) Keeping in mind that $\boldsymbol{\tau}$ matrices do not commute, is $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_1$? [Hint: be clear about what spaces these act in!]

   (c) Why do we not consider terms in the potential of the form $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^n$ for $n > 1$ or $(\boldsymbol{L} \cdot \boldsymbol{S})^n$ for $n > 2$?

   (d) What are the electromagnetic interactions for np and nn scattering?

   (e) Add to the list of physical situations where a parameter is taken either to 0 or infinity to simplify a problem (and then one can expand around this limit).

   (f) You will often hear that effective (field) theories exploit a "separation of scales". What does this mean?

   (g) How high in energy do we need to know NN phase shifts to do nuclear structure?

   (h) Why should a strong repulsive short-range potential make (some) many-body calculations of nuclei more difficult? Would a mean-field approximation be accurate?

   (i) You often hear it said that the problem must be nonperturbative because there are bound states. Why can't you find bound states in perturbation theory?

   (j) Why is the deuteron bound but two neutrons are not bound? Why are two protons not bound?

2. Basic skills.

   (a) Verify:
   $$e^{i\widehat{S}}\widehat{O}e^{-i\widehat{S}} = \widehat{O} + i[\widehat{S},\widehat{O}] + \frac{i^2}{2!}[\widehat{S},[\widehat{S},\widehat{O}]] + \frac{i^3}{3!}[\widehat{S},[\widehat{S},[\widehat{S},\widehat{O}]]] + \cdots$$
   explicitly to this order (or until you run out of patience :).

   (b) If $U = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{G}}$ is a symmetry operation, show that it is necessary and sufficient that $[\boldsymbol{G},\widehat{O}] = 0$ for an operator $\widehat{O}$ to be invariant under the symmetry.

3. Devise a way to estimate the range of the parts of the NN interaction from pion exchange, "σ" exchange (which is generated by correlated two-pion exchange at a mass of about 500 MeV), and ω exchange (look up the mass online if necessary). Are your results consistent with the pictures of the $^1\mathrm{S}_0$ potentials in the lecture notes?

4. The usual plot you see of the central nucleon-nucleon (NN) potential is in the $^1\mathrm{S}_0$ channel (e.g., in the figures shown in the lectures). The notation is $^{2S+1}L_J$, with $S = 0$ (singlet) or $S = 1$ (triplet), $L = 0, 1, 2, \ldots$ (with the corresponding letter), and J takes on values consistent with L and S.

   (a) What are the possible channels for $L = 0$, 1, and 2 for neutron-proton scattering?

   (b) Same question, but for neutron-neutron scattering. (Hint: nucleons are fermions, so the total two-particle wave function must be antisymmetric.)

   (c) Experimental: What are the difficulties and advantages of scattering neutrons from proton targets versus protons from neutron targets?

5. Estimate the radius and energy of hydrogen-like atoms using dimensional analysis.

6. We claim the following hierarchy of (three) scales for a hydrogen atom:

   - electron mass $m_e \approx 0.511\,\mathrm{MeV}$
   - characteristic momentum $p \sim \alpha m_e \approx 3.6\,\mathrm{keV}$
   - characteristic energy $B \sim \frac{1}{2}\alpha^2 m_e \approx 13.6\,\mathrm{eV}$

   Derive the scaling with α and m_e by simple scaling arguments (that could be applied to other systems). In particular,

   (a) Apply the uncertainty relation to relate the momentum p to the characteristic radius R (in this case it is the Bohr radius, but we pretend we don't know it yet).

   (b) Use this to eliminate p from the total energy (sum of kinetic and potential) to find $E(R)$.

   (c) Minimize $E(R)$ to find R and therefore p and then the value at the minimum, verifying the results quoted above.

   (d) For this example, why is there a hierarchy? Would there be a hierarchy of the same type in QCD with the strong coupling α_s instead of the fine structure constant?

   (e) What is the analogous hierarchy exploited in chiral (i.e., pionful) effective field theory?

7. [Advanced] Deriving the Coulomb potential from QED by actually integrating out the photon field. (This is an alternative to matching QED and potential calculations of scattering, which we could also do.) Consider the QED Lagrangian including gauge-fixing:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QED}} &= \frac{1}{2}A_\mu[g^{\mu\nu}\partial_\lambda\partial^\lambda - (\xi^{-1}-1)\partial^\mu\partial^\nu]A_\nu - j_e^\mu A_\mu + \overline{\psi}(i\partial\!\!\!/ - m)\psi \\
&= \frac{1}{2}A_\mu[D_{\mathrm{F}}^{\mu\nu}]^{-1}A_\nu - j_e^\mu A_\mu + \overline{\psi}(i\partial\!\!\!/ - m)\psi \,,
\end{aligned}
$$

   with electromagnetic current (charge e) $j_e^\mu = e\overline{\psi}\gamma^\mu\psi$. (The second line defines D_{F} as the inverse of the operator in the first line, where F means to use Feynman boundary conditions.) The physics of electrons and photons can be derived (e.g., Feynman diagrams) from the functional (path) integral:

$$Z = \int \mathcal{D}\overline{\psi}\, \mathcal{D}\psi\, \mathcal{D}A\, \exp[iS(\overline{\psi}, \psi, A)]$$

(after adding external sources, which we omit here). But suppose we "integrate out" the photon field A_μ (which we can do because it appears at most quadratically):

$$\exp[iS_{\text{eff}}(\overline{\psi}, \psi)] = \int \mathcal{D}A\, \exp[iS(\overline{\psi}, \psi, A)]\ .$$

(a) Complete the square to show (with $j_e^\mu = e\overline{\psi}\gamma^\mu\psi$)

$$S_{\text{eff}} = \int d^4x\, \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) + \frac{1}{2}\int d^4x\, d^4y\, j_e^\mu(x) D_{\text{F}\mu\nu}(x-y) j_e^\nu(y)\ ,$$

where

$$[g_{\mu\nu}\partial_\lambda\partial^\lambda - (\xi^{-1} - 1)\partial_\mu\partial_\nu] D_{\text{F}}^{\nu\rho}(x-y) = i\delta_\mu{}^\rho \delta^{(4)}(x-y)$$

or (after a Fourier transform)

$$[-k^2 g_{\mu\nu} + (1 - \frac{1}{\xi})k_\mu k_\nu] D_{\text{F}}^{\nu\rho}(k) = i\delta_\mu{}^\rho$$

which leads to (check that this works!)

$$D_{\text{F}}^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)\ .$$

(b) Can we directly identify the last term in S_{eff} as (with particle density $\rho = \psi^\dagger\psi$)

$$-\frac{1}{2}\int dt \int d^3x\, d^3y\, \rho(\mathbf{x}, t) V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t)$$

and in doing so identify the potential V? (Think a bit about it but then come back to this part after doing the next two sections.)

(c) If we consider a classical *static* distribution $j_e^\mu \to e(\rho, \mathbf{0})$, show that

$$V(\mathbf{x} - \mathbf{y}) = -e^2 \int dt' \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik^\mu(x-y)_\mu}}{k_0^2 - \mathbf{k}^2 + i\epsilon} = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = \frac{e^2}{4\pi|\mathbf{x} - \mathbf{y}|}\ ,$$

which is the Coulomb interaction. Is making the distribution static a big assumption?

(d) Really the current density is a quantum-mechanical operator. What does this imply for defining $V(\mathbf{x} - \mathbf{y})$ quantum mechanically? Where are the ambiguities in defining V?

(e) Suppose we were exchanging a massive boson instead of a massless photon? How would the derivation change? Do we always get a local potential?