# Physics 8805: Nuclear Few- and Many-Body Physics
# Discussion questions and exercises for Scattering 1

[Last revised on September 5, 2014 at 01:01:30.]

1. Scattering review (basic skills and two-minute and (some) discussion questions):

   (a) What do "on-shell" and "off-shell" mean in the context of scattering?

   (b) Under what conditions is a partial-wave expansion of the potential useful?

   (c) Why is it valid and useful for two-body scattering to switch to relative and center-of-mass coordinates?

   (d) Discuss why a classical potential energy is measurable but a quantum mechanical potential is generally not a measurable quantity. When *is* a potential measurable? (Possible hint: Remember that we discussed a calculated potential between quarks in QCD 1; what made it possible to define a potential in that case.)

   (e) Derive the standard result:

   $$\frac{e^{i\delta_l(k)}\sin\delta_l(k)}{k} = \frac{1}{k\cot\delta_l(k) - ik}$$

   [Hint 1: First move $e^{i\delta_l}$ to the denominator, then replace it by $\cos + i\sin$.]

   (f) Given a potential that is not identically zero as $r \to \infty$ (e.g., a Yukawa), how would you know in practice where the asymptotic (large $r$) region starts?

   (g) What is the physical interpretation of the relation between the (partial-wave) S-matrix and the scattering amplitude? (Note that $S_l(k) = 1 + 2ikf_l(k)$.)

   (h) How can we describe scattering using plane waves when a particle described by a plane wave has equal amplitude to be anywhere?

   (i) Advanced: what is the analytic structure of the T-matrix near a bound-state energy?

2. Two-minute questions on unitary transformations:

   (a) If we transform eigenstates by a unitary transformation $|\widetilde{\psi}\rangle = \widehat{U}|\psi\rangle$, how must an arbitrary operator $\widehat{O}$ transform so that its matrix elements between unitarily transformed eigenstates are unchanged?

   (b) What properties of a bound state will change under a (short-range) unitary transformation and what will be unchanged? Consider (and justify your answer)

      i. the bound-state energy

      ii. the wave function (bound or scattering) beyond the range of the potential

      iii. the wave function within the potential

      iv. the expectation value of the radius squared

3. Exploring the Lippmann-Schwinger equation. [The conventions here follow Taylor.]

(a) Using the Schrödinger equation for the scattering of two particles with mass $m$,

$$(H_0 + V)|\psi_E\rangle = E|\psi_E\rangle \ ,$$

where $H_0$ is the free Hamiltonian, show that the Lippmann-Schwinger equation for the wave function,

$$|\psi_E^{\pm}\rangle = |\phi_k\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\psi_E^{\pm}\rangle \ ,$$

is satisfied. Here $E = k^2/m$ and the plane wave state satisfies $H_0|\phi_k\rangle = E|\phi_k\rangle$. Why do you need the $\pm i\epsilon$?

(b) We can define the $T$-matrix on-shell as the transition matrix that acting on the plane wave state yields the same result as the potential acting on the full scattering state. That is, $T^{(\pm)}(E = k^2/m)|\phi_k\rangle = V|\psi_E^{\pm}\rangle$. What does it mean that the $T$-matrix is "on-shell"? (This is a really quick question!)

(c) Show that matrix elements of the $T$-matrix satisfy the Lippmann-Schwinger equation

$$\langle \mathbf{k}'|T^{(\pm)}(E)|\mathbf{k}\rangle = \langle \mathbf{k}'|V|\mathbf{k}\rangle + \int d^3p \, \frac{\langle \mathbf{k}'|V|\mathbf{p}\rangle\langle \mathbf{p}|T^{(\pm)}(E)|\mathbf{k}\rangle}{E - \frac{p^2}{m} \pm i\epsilon} \ .$$

What normalization is used for the momentum states? [See the Morrison and A.N. Feldt pedagogical article under Program→References on the webpage.] Are the matrix elements of the $T$-matrix on the right side on-shell?

(d) Write the Lippmann-Schwinger equation for the wave function in coordinate space for a local potential $V = V(\mathbf{r})$. To this end, show first that the free Green's function

$$G^{\pm}(\mathbf{r}', \mathbf{r}; E = k^2/m) = \langle \mathbf{r}|\frac{1}{E - H_0 \pm i\epsilon}|\mathbf{r}'\rangle$$

is given by

$$G^{\pm}(\mathbf{r}', \mathbf{r}; E = k^2/m) = -\frac{m}{4\pi}\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \ .$$

(e) Show that when the $T$-matrix is evaluated on-shell, it is proportional to the scattering amplitude, $T^{+}(E = k^2/m) = -\frac{1}{4\pi^2 m} f(k, \theta)$, by analyzing the asymptotic form of the Lippmann-Schwinger equation and comparing to

$$\langle \mathbf{r}|\psi_E^{+}\rangle \overset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(k,\theta)\frac{e^{ikr}}{r}\right) \ .$$

(f) Start from the momentum-space partial wave expansion of the potential,

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = \frac{2}{\pi}\sum_{l,m} V_l(k', k) Y^*_{lm}(\Omega_{k'}) Y_{lm}(\Omega_k)$$

and a similar expansion of the $T$-matrix to derive the partial wave version of the Lippmann-Schwinger equation (with the correct factor for the integral):

$$T_l(k', k; E) = V_l(k', k) + \frac{2}{\pi}\int_0^\infty dp\, p^2 \frac{V_l(k', q) T_l(q, k; E)}{E - p^2/m + i\epsilon} \ .$$

4. Consider two momentum-space potentials, $V_1(\mathbf{k}, \mathbf{k}') = V_0\, e^{-(k^2+k'^2)/\mu^2}$ and $V_2(\mathbf{k}, \mathbf{k}') = V_0\, e^{-(\mathbf{k}-\mathbf{k}')^2/\mu^2}$.

   (a) Are they local or non-local?

   (b) Do they have P-wave projections? (That is, if you wrote it in the partial-wave expansion would there be an $L = 1$ term?)

   (c) Do they have higher angular momentum projections?

5. Show directly from the Fourier transform expression of a local potential, without specifying its functional form, that the momentum space version will only depend on the momentum transfer $\mathbf{k}' - \mathbf{k}$.

6. Scattering phase shifts for a square well potential.

   (a) Calculate the S-wave scattering phase shifts for an attractive square-well potential $V(r) = -V_0\theta(R - r)$ and show that

$$\delta_0(E) = \arctan\left[\sqrt{\frac{E}{E+V_0}}\tan(R\sqrt{2\mu(E+V_0)})\right] - R\sqrt{2\mu E}$$

   (b) Let's consider the analytic structure of the corresponding partial-wave S matrix, which is given by

$$S_0(k) = e^{-2ikR}\frac{k_0\cot k_0 R + ik}{k_0\cot k_0 R - ik}$$

   where $E = k^2/2\mu$ and $k_0^2 = k^2 + 2\mu V_0$.

      i. Show that $S_l(k) = e^{2i\delta_l(k)}$ for $l = 0$ is satisfied. [Hint: write $e^{2i\delta} = e^{i\delta}/e^{-i\delta}$.]

      ii. Treat $S_0(k)$ as a function of the complex variable $k$ and find its singularities.

      iii. Bound states are associated with poles on the imaginary axis in the upper half plane. Show that the condition for such a pole here gives the same eigenvalue condition (a transcendental equation) that you would get from a conventional solution to the square well by matching logarithmic derivatives. [Define $k = i\kappa$ with $\kappa > 0$ when analyzing such a pole.]

7. Variable phase approach (VPA) for finding phase shifts from a local potential. Here we consider s-waves. [References: Taylor, *Scattering Theory*, pp. 197-201, Calogero, *The Variable Phase Approach to Potential Scattering*, (Academic Press, New York, 1967).]

   (a) Define the truncated potential $V_\rho(r)$ by

$$V_\rho(r) = V(r)\theta(\rho - r)\ .$$

   That is, it is the usual potential for $r \le \rho$, but identically zero beyond that. Then we define $\delta(k, \rho)$ as the phase shift for $V_\rho$ at momentum $k$. The phase shift we want is

$\delta(k) = \lim_{\rho\to\infty} \delta(k,\rho)$. The basis of the variable phase method is a differential equation for $\delta(k,r)$ at fixed $k$ (again, this is the s-wave equation):

$$\frac{d\delta(k,r)}{dr} = -\frac{1}{k} 2MV(r)\sin^2[kr + \delta(k,r)] \; ,$$

which is a nonlinear first-order differential equation with initial condition $\delta(k,0) = 0$. Think about how you would implement this in your favorite programming language.

(b) The Mathematica notebook `square_well_scattering.nb` implements the VPA for a square well. Changing to a different potential is trivial (see the illustration at the end with a combined short-range repulsive square well and a mid-range attractive square well). Show that it reproduces the known phase shifts for the square well result.

(c) (Optional) The derivation of the VPA is outlined in the lecture notes. Fill in the details and/or generalize to arbitrary (uncoupled) $l$.

(d) Show from the VPA differential equation that a fully attractive potential gives a positive phase shift and a fully negative potential gives a negative phase shift.

(e) The VPA automatically builds in Levinson's theorem about the number of bound states and the phase shift at zero. How? [Hint: what is the condition imposed on the phase shift at large energy for Levinson's theorem? Consider integrating $d\delta(k,r)/dr$ in $r$ from zero to infinity. Use $\sin^2 x \leq 1$ to put a bound on $\delta(k)$.]

(f) Things to try numerically with the Mathematica or Python notebooks:

- Try out Levinson's theorem in practice (e.g., for a square well where the number of bound states versus depth is easily found in parallel).
- Explore the effective range expansion by looking at $k \cot\delta(k)$ at small $k$ and extracting parameters or verifying the connection to bound-state properties.
- (Advanced) Do a Lepage plot exercise.
- (Advanced) Check that a sample unitary transformation does not change the phase shifts. To use our VPA implementation for this exercise we'll need to restrict attention to transformations that don't introduce non-localities into the potential. [Look up the UCOM potential for ideas on how to build such a transformation.]

(g) (Advanced) Is it possible to generalize the LPA to coupled channels and non-local potentials? Yes! Think about how to do it, but probably just check your ideas against the references.

8. Numerically solving the partial-wave Lippmann-Schwinger equation in momentum space based on the discussion in Landau's Quantum Mechanics II book, Section 18.3. You should follow along with one of the sample implementations of this procedure, even if you don't fill in all the details. The discussion applies directly to uncoupled channels; we can discuss the extension to coupled channels if there is interest.

(a) We will solve for what Landau calls the R-matrix (known as the K-matrix in other contexts or sometimes also the T-matrix despite the boundary conditions). The Lippmann-Schwinger (LS) equation for $R_l$ is

$$R_l(k',k;E) = V_l(k',k) + \frac{2}{\pi}\mathbb{P}\int_0^\infty dq\, \frac{q^2 V_l(k',q)R_l(q,k;E)}{E-E_q} \ ,$$

where $E_k \equiv k^2/2\mu$, $\mu = m/2$, and we work in units where $\hbar^2/m = 1$ ($= 41.47105\,\text{MeV-fm}^2$ for $np$). We get the desired phase shift from

$$R_l(k_0,k_0;E_{k_0}) = -\frac{\tan\delta_l(k_0)}{2\mu k_0} = -\frac{1}{2\mu}\frac{1}{k_0\cot\delta_l(k_0)} \ .$$

The usual LS equation for the T-matrix for outgoing boundary conditions has $+i\epsilon$ in the denominator. What kind of boundary conditions are the ones here? (Hint: the principal value is half the sum of incoming and outgoing waves.) Why might we prefer to solve this equation numerically instead of the one for the T-matrix?

(b) We want to evaluate the integral equation on a discrete mesh of momenta, but we need to deal with the principal value. The "trick" is to add and subtract to the integral (we've now explictly set $\hbar^2/m = 1$):

$$\begin{aligned}\frac{2}{\pi}\mathbb{P}\int_0^\infty dq\, \frac{q^2 V_l(k',q)R_l(q,k;k_0^2)}{k_0^2-q^2} \ &= \ \frac{2}{\pi}\int_0^\infty dq\, \frac{q^2 V_l(k',q)R_l(q,k;k_0^2) - k_0^2 V_l(k',k_0)R_l(k_0,k)}{k_0^2-q^2} \\ &\quad + \frac{2}{\pi}k_0^2 V_l(k',k_0)R_l(k_0,k,k_0^2)\,\mathbb{P}\int_0^\infty dq\, \frac{1}{k_0^2-q^2} \ .\end{aligned}$$

Why can we remove the $\mathbb{P}$ from the first term? Show that the integral in the second term is zero. What would it be if the integrals were up to a cutoff $\Lambda$ instead of $\infty$?

(c) Now we can solve the integral equation numerically by replacing the continuous momentum in the integral by a set of $N$ discrete momenta $\{k_i\}$ and weights $\{w_i\}$, $i = 1, N$, that correspond to gaussian quadrature points and weights. We also define $k_{N+1} = k_0$. Show that the LS equation for $(N+1)\times(N+1)$ matrix $R$ becomes

$$R_{ij} = V_{ij} + \frac{2}{\pi}\sum_{l=1}^{N}\frac{k_l^2 V_{il}R_{lj}w_l}{k_0^2-k_l^2} - \frac{2}{\pi}\left(\sum_{l=1}^{N}\frac{w_l}{k_0^2-k_l^2}\right)k_0^2 V_{i,N+1}R_{N+1,j} \ .$$

(d) Show that this can be written as

$$R_{ij} + \sum_{l=1}^{N+1} V_{il}D_l R_{lj} = V_{ij} \ ,$$

where

$$D_i \equiv \begin{cases} \frac{2}{\pi}\frac{w_i k_i^2}{k_i^2-k_0^2} & i = 1, N \\ -\frac{2}{\pi}k_0^2\left(\sum_{l=1}^{N}\frac{w_l}{k_l^2-k_0^2}\right) & i = N+1 \end{cases}$$

We see that this can be written as a matrix equation (with implied sum over $l$):

$$(\delta_{il} + V_{il}D_l)\,R_{lj} \equiv F_{il}R_{lj} = V_{ij} \ ,$$

which can be solved to find $R = F^{-1}V$. Then the matrix element we want is $R_{N+1\,N+1}$.

9. More on the Lippmann-Schwinger (LS) equation.

(a) In the "Exploring the LS equation" problem we used the momentum space matrix elements of the operator LS equation (we omit the hats here):

$$T^{(\pm)}(E) = V + V\frac{1}{E - H_0 \pm i\epsilon}T^{(\pm)}(E) \; .$$

Show that this can also be written as

$$T^{(\pm)}(E) = V + V\frac{1}{E - H \pm i\epsilon}V \; ,$$

where now the full Green's function appears (it has $H$ instead of $H_0$). Do this by repeating the derivation but now using the alternative LS equation for the wave function (show that it works!):

$$|\psi_E^{\pm}\rangle = |\phi_k\rangle + \frac{1}{E - H \pm i\epsilon}V|\phi_k\rangle \; .$$

(b) Now use the "spectral representation"

$$\frac{1}{E - H \pm i\epsilon} = \sum_n \frac{|\psi_n\rangle\langle\psi_n|}{E - E_n} + \int d^3p \, \frac{|\psi_p^+\rangle\langle\psi_p^+|}{E - p^2/m \pm i\epsilon} \; ,$$

which follows by inserting a complete set of bound and scattering eigenstates of $H$, to show that as a function of energy $E$, the momentum-space $T$-matrix has simple poles at the bound-state energies $E_n$ with separable residues $\langle\mathbf{k}'|V|\psi_n\rangle\langle\psi_n|V|\mathbf{k}\rangle$.