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Chapter 1

Introduction to QCD and the Standard Model

It is an astounding fact that modern-day physics can quantitatively describe phenomena from the small scale of quarks and leptons (10^{-18} cm) to the large scale of the whole present-day Universe (10^{28} cm) using the four fundamental forces: gravity, electromagnetism, strong and weak interactions. While gravity is governed by Einstein's general relativity, the other three forces can be described to an excellent degree by a quantum field theory of quarks and leptons based on a framework consistent with Einstein's special theory of relativity and quantum mechanics: the so-called standard model (SM). This book is mainly about the physics of *strong interactions*, the associated structure and interactions of hadronic matter, and its role in the evolution of the Universe from a very early stage (microseconds young) to the present day (13 billion years old). However, as the reader will discover, it is not possible to tell the story without including the other forces. Therefore in this beginning chapter, we give an introduction to the standard model, with an emphasis on the fundamental theory of strong interactions—quantum chromodynamics (QCD).

1.1 Quarks

The quarks will be major characters in our story and are the fundamental objects participating in strong interactions. Like the electrons, they are simple structureless (as far as we know) spin-1/2 particles. Therefore in a relativistic theory, they are described by Dirac spinors $\psi_\alpha(x)$ with four components $\alpha = 1, ..., 4$, which are functions of the space-time coordinates $x^\mu = (t, x, y, z)$. If quarks did not interact with other particles or fields, they would obey the free Dirac equation,

$$(i \not\partial - m)\psi(x) = 0 \ , \tag{1.1}$$

where m is meant to be the "free" mass and other notations follow from the standard textbooks like Bjorken and Drell, or Peskin and Schroder. Simple states of these quarks are the standard plane waves

$$\psi_{k,\lambda}(x) = u(k, \lambda)e^{-i(Et-\vec{x}\cdot\vec{k})} \ , \tag{1.2}$$

where $k^\mu = (E, \vec{k})$ and λ denote the four-momentum and polarization, and $u(k, \lambda)$ is the spin-dependent momentum-space wave function. We note that the Dirac equation can also be derived from the lagrangian density,

$$\mathcal{L}(x) = \bar{\psi}(x)(i \not{\partial} - m)\psi(x) \ . \tag{1.3}$$

In reality, however, free quarks have never been observed in laboratory; this is known as *quark confinement*, and is an important consequence of the low-energy dynamics of strong interactions. We will discuss this phenomenon more extensively below and in subsequent chapters.

At present, we have discovered six flavors of quarks through various high-energy experiments: up(u), down(d), charm(c), strange(s), top(t) and bottom(b). These flavors organize themselves into families under the weak interactions. Up and down form the first family (or generation), charm and strange the second, and top and bottom the third. As far as we can ascertain, these three families are basically repetitions of the same pattern (same quantum numbers) with unknown physical significance. For example, the electric charges of the up, charm and top quarks are all 2/3 of that of the proton, and the charges of the down, strange, and bottom are all −1/3. They are distinguished by their masses and associated "flavor" quantum numbers.

Quark Masses

Because we cannot observe free quarks, their masses obviously cannot be measured directly and so the very meaning of mass requires some explanation. The *mass of a quark* is actually just a parameter in the lagrangian of the theory, which describes the self-interaction of the quark, and is not directly observable. As such, the mass parameter is much like a coupling constant in quantum field theory, and is technically dependent on the momentum scale and the renormalization scheme and scale-dependent. (In this book, unless specified otherwise, we always use the so-called *dimensional regularization and modified minimal subtraction*, or in short $\overline{\text{MS}}$ quark masses.)

According to the standard model, the masses of the quarks are generated through a symmetry breaking phase transition of the electroweak interactions (a transition similar to that of a normal conductor to superconductor in condensed matter physics, in which an effective mass for the photon is produced). The detailed aspects of the symmetry breaking, such as the existence of Higgs bosons, are still under investigation in experiments at high-energy colliders. Our present knowledge of the quark masses is shown in Table 1.1

Quark Flavor:	up	down	charm	strange	top	bottom
Mass:	1.5-4 MeV	4-8 MeV	1.25 GeV	$\sim$ 100 MeV	175 GeV	4.25 GeV
Charge:	2/3	−1/3	2/3	−1/3	2/3	−1/3

Table 1.1: Quark masses in the $\overline{\text{MS}}$ renormalization scheme at a scale of $\mu = 2$ GeV.

Color Charge

The quarks participate in strong interactions because they carry *color charges.* The color charges are the analogs of electric charge in Quantum Electrodynamics, but with important differences. Unlike electric charge, which is a scalar quantity in the sense that the total charge of an electric system is simply the algebraic sum of individual charges, the color charge is a *quantum vector* charge, a concept similar to angular momentum in quantum mechanics. The total color charge of a system must be obtained by combining the individual charges of the constituents according to group theoretic rules analogous to those for combining angular momenta in quantum mechanics.

The quarks have three basic color-charge states, which can be labeled as $i = 1, 2, 3$, or red, green, and blue, mimicking three fundamental colors. Three color states form a basis in a 3-dimensional complex vector space. A general color state of a quark is then a vector in this space. The color state can be rotated by 3×3 unitary matrices. All such unitary transformations with unit determinant form a Lie group SU(3). The 3-dimensional color space forms a fundamental representation of SU(3). It is customary to label the color charges by the spaces of the SU(3) representations, **3** in the case of a quark. The rules of adding together color charges follow those of adding *representation* spaces of the SU(3) group, which are straightforward extensions of adding up angular momenta in quantum mechanics. Some examples of adding up quark charges can be found in in the Appendix of this chapter.

The quarks, like the electron, have anti-particles, called antiquarks, often denoted by $\bar{q}$. The antiquarks have the same spin and mass as the quarks, but with opposite electric charges. For example, an anti-up quark has an electric charge $-2/3$ of the proton charge. The color charge of an antiquark is denoted $\bar{\mathbf{3}}$, which is a representation space of SU(3) where the vectors are transformed according to the complex conjugate of an SU(3) matrix.

One can more generally view quark confinement as color confinement: strong interactions do not allow states other than color-singlet, or color-neutral to appear in nature. There is strong evidence for color confinement in numerical simulations of QCD, known as lattice QCD calculations. However, there is no rigorous mathematical proof in the literature to date.

1.2 Gluons and Quantum Chromodynamics

Quarks do not interact with each other directly; they do so through intermediate agents called *gluons*. A simple way to understand this is that the gluons in strong interactions play the role of photons in quantum electrodynamics (QED), which mediate electromagnetic interactions between charged currents. Like photons, gluons are massless, spin-1 particles with two polarization states (left-handed and right-handed). They are represented by a four-component vector potential $A^\mu(x)$ with a Lorentz index $\mu = 0, 1, 2, 3$, just as in electromagnetism. Therefore, conditions must be imposed on $A^\mu(x)$ to select only the physical degrees of freedom, as different $A^\mu(x)$ can give rise to the same physics. These conditions are called *gauge conditions* or gauge choices. For the same reason, $A^\mu(x)$ are called *gauge potentials*, and gluons are called *gauge particles*.

Although there is only one type of photon mediating electromagnetic interactions, there are 8 types of gluons mediating the strong interactions. This proliferation of gauge particles has to do with the SU(3) color symmetry: Return to the free quark lagrangian density in Eq. (1.2), where the quark field is now endowed with a new index i for color. It is easy to see that $\mathcal{L}$ is invariant under a global SU(3) transformation

$$\psi'(x) = U\psi(x) \ , \tag{1.4}$$

where U is a 3×3 unitary matrix acting on color index, and"global" means that the field at different spacetime is transformed in exactly the same way. A generic SU(3) matrix requires 8 real parameters, usually written in the form

$$U = \exp(i \sum_a \theta^a \lambda^a / 2) \ , \tag{1.5}$$

where $\lambda^a/2$ ($a = 1, ..., 8$) are 3×3 hermitian matrices and are the so-called generators of SU(3) rotations, just like the angular momentum operators L_i being generators of ordinary space rotations. A commonly used definition of these matrices is due to Gell-Mann (and hence named after him) and can be found in the Appendix.

If we introduce 8 gluon potentials, A^μ_a, as well as the associated covariant derivative, $D^\mu \equiv \partial^\mu + igA^\mu_a\lambda^a/2$, the free quark lagrangian can be modified to

$$\mathcal{L}_q = \bar{\psi}(i \not{D} - m_q)\psi \ . \tag{1.6}$$

The new lagrangian has a remarkable symmetry called *gauge symmetry*: It is invariant under a *spacetime-dependent* SU(3) rotation $U(x)$ of the quark fields if, at the same time, the gauge potentials transform according to

$$A^\mu \to U(A^\mu + \frac{i}{g}\partial^\mu)U^\dagger \ , \tag{1.7}$$

where we have introduced the 3×3 gluon potential matrix $A^\mu = A^\mu_a\lambda^a/2$. The above transformation is a generalization of gauge transformations in classical electromagnetism, and reflects that the number of physical degrees of freedom associated with each gauge potential is just 2, those of massless spin-1 particles!

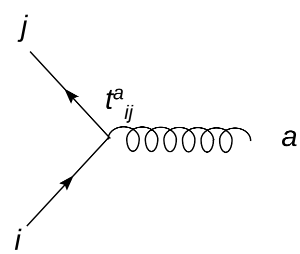

Figure 1.1: The color of a quark can change from i to j by a gluon of color a, coupled through SU(3) generator $t^a_{ij} = \lambda^a_{ij}/2$.

Therefore, by requiring a local color (gauge) symmetry, we find that the quarks are no longer free particles, but they interact with each other through the new gauge potentials. In other words, the gauge symmetry has generated a well-defined dynamics of the color charges. The 8 gluons are simply related to the 8 parameters of a general SU(3) transformation!

Let us examine these color interactions in some details. The interaction term in the lagrangian is

$$\mathcal{L}_{\rm int} = -g\bar{\psi}_j A^\mu_a \frac{\lambda^a_{ji}}{2}\gamma_\mu\psi_i \ , \tag{1.8}$$

where g is the coupling constant. Therefore, quarks interact with gluons in a way similar to electrons interacting with photons. A new feature here is that the quark can change its color from i to j by emitting or absorbing a gluon of color a, coupling through an SU(3) generator $t^a_{ij} = \lambda^a_{ij}/2$, as shown in Fig. 1.1.

Gluons are physical degrees of freedom and therefore must carry energy and momentum themselves. Thus one must add additional terms in the lagrangian to describe these physical features.

Following the successful theory of Maxwell on electromagnetism, we introduce in a similar way the antisymmetric field strength tensor $F^a_{\mu\nu}$ and the kinetic energy term for the gluons

$$\mathcal{L}_g = -\frac{1}{4}F^{\mu\nu a}F_{\mu\nu a} \tag{1.9}$$

where summation over color is implicit. To ensure the new term added is gauge-invariant, the field strength tensor is required to transform according to $F^{\mu\nu} \to UF^{\mu\nu}U^\dagger$. This can be accomplished by choosing $F^{\mu\nu} = -ig[D^\mu, D^\nu]$, or

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - g[A^\mu, A^\nu] \ , \tag{1.10}$$

where apart from the standard partial derivative terms, there is a commutator term $[A^\mu, A^\nu]$. This is non-linear in terms of the gauge potential. Therefore, the eight gluons do not come in as a simple repetition of the photon in QED—they have three- and four-gluon self-interactions! We depict such interaction terms schematically in Fig. 1.2.

Figure 1.2: Self-interactions of gluons which are responsible for many important features of QCD, such as asymptotic freedom, color confinement and chiral symmetry breaking.

The full QCD lagrangian density is the sum of quark and gluon terms,

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm q} + \mathcal{L}_{\rm g} \ . \tag{1.11}$$

Because the gluons carry color charges, their self-interactions are a source of the key differences between QCD and QED. In fact, these interactions are responsible for many of the unique and salient features of QCD, such as asymptotic freedom, chiral symmetry breaking, and color confinement. In the remainder of this chapter, we briefly discuss these important properties of color dynamics.

1.3 Force Between Quarks, Asymptotic Freedom

A first exercise in studying electromagnetic physics is to consider the force between two electric charges. Similarly, a starting point to understand strong interactions is to consider the strong force between two quarks. Of course, we forget for the moment the fact that the free quarks do not exist and the interaction between quarks may not be calculable through simple one-gluon exchange. We press ahead anyway just to see what follows.

As shown by Feynman diagram in Fig. 1.3, a quark of color i, exchanging a gluon with another quark of color j, scatters into a quark of color i', along with a quark of color j'. The scattering amplitude for this process is

$$S \sim (-igt^a_{i'i}\gamma^\mu)D_{\mu\nu}(q)(-igt^a_{j'j}\gamma^\nu) \ , \tag{1.12}$$

where we have taken into account all 8 gluon exchanges by summing $a = 1, ..., 8$. The four-momentum q of the gluon is space-like, $q^2 = q_0^2 - \vec{q}^2 < 0$. $D_{\mu\nu}(q)$ is the gluon propagator in Feynman gauge (a particularly convenient gauge choice!),

$$D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2} \ . \tag{1.13}$$

The only difference between the above amplitude and that for two electrons is the color factor $t^a_{i'i}t^a_{j'j}$.

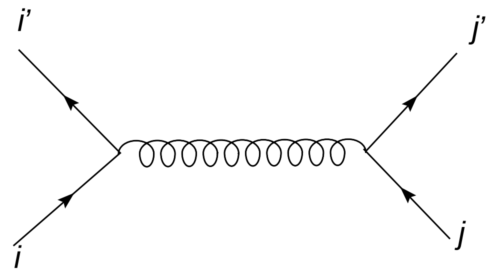

Figure 1.3: Strong interactions of two quarks through one-gluon exchange.

Let us calculate the color factors for the two quarks in **6** and $\bar{\mathbf{3}}$, respectively. If two quarks are in **6**, for example, we can take a specific case $i = j = i' = j' = 1$. By summing over a, the average color factor is 1/3. Thus the interaction between the quarks in the symmetric color state is repulsive. This sign is analogous to the electric force. On the other hand, if the two quarks are in $\bar{\mathbf{3}}$, the color factor is $-2/3$, and the force between them is attractive! Therefore we conclude that the force between two quarks depend on the color states! For a more detailed discussion on this, see Halzen and Martin.

Exercise: work out the sign of color force between a quark and antiquark when they are in 1 and 8, respectively.

Asymptotic Freedom

One of the striking properties of QCD is *asymptotic freedom* which states that the interaction strength $\alpha_s = g^2/4\pi$ between quarks becomes smaller as the distance between them gets shorter. For this observation, Gross, Politzer and Wilczek won 2004 Nobel prize in physics.

To explain asymptotic freedom, let us first recall that in electrodynamics, the force between two charges q_1 and q_2 in vacuum is described by Coulomb's law,

$$F = \frac{1}{4\pi}\frac{q_1 q_2}{r^2} \ . \tag{1.14}$$

If, on the other hand, the two charges are submerged in a medium with dieletric constant $\epsilon > 1$ (charge screening), the force becomes

$$F = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{r^2} \ , \tag{1.15}$$

which also can be expressed in the above vacuum form by introducing the effective charge $\tilde{q}_i = q_i/\sqrt{\epsilon}$. Thus the effect of the medium may be regarded as modifying the charges.

In the quantum field theory, the vacuum is not empty because it is just the lowest energy state of a field system and is filled with electrons of negative energies from one point of view. When a photon passes through the vacuum, it can induce transitions of an electron from negative to positive energy states, virtually creating a pair of electron and positron, known as vacuum fluctuation, shown in the first diagram in Fig. 1.4. Because of this, the interaction between two electrons in the vacuum becomes

$$F = \frac{e_{\rm eff}^2}{4\pi r^2} = \frac{\alpha_{\rm em}(r)}{r^2} \ , \tag{1.16}$$

where $\alpha_{\rm em}$ is an effective fine structure constant, depending on the distance r, or momentum transfer $q \sim 1/r$. As $r \to \infty$ or equivalently, $q \to 0$, the coupling measured the interaction strength of the low-energy photon, and is what often quoted, $\alpha_{\rm em}(q=0) = 1/137.035$.

The dependence of the fine structure constant on the distance or momentum scale can be determined by a differential equation in QED

$$\mu \frac{d\alpha(\mu)}{d\mu} = \beta(\alpha(\mu)) \ , \tag{1.17}$$

where μ is a momentum scale, roughly corresponding to $1/r$. The above is also an example of renormalization group equations. The β-function may be calculated in perturbation theory because $\alpha_{\rm em}$ is small and at one loop order $\beta = 2\alpha_{\rm em}^2/3\pi > 0$. The solution is thus

$$\alpha_{\rm em}(\mu) = \frac{\alpha_{\rm em}(\mu_0)}{1 - \frac{\alpha_{\rm em}(\mu_0)}{3\pi} \ln \frac{\mu^2}{\mu_0^2}} \ . \tag{1.18}$$

Note that the interaction strength of the two electrons gets stronger as the distance between them becomes smaller. Therefore, QED becomes a strongly-coupled theory at very short distance scale. For this reason, it cannot be solved in a completely consistent way unless an ultraviolet momentum cut-off is imposed on the theory.

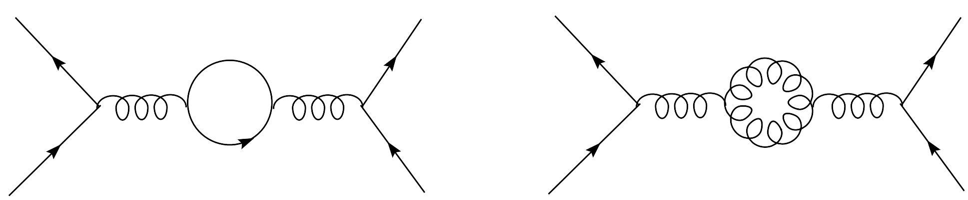

Figure 1.4: The quantum vacuum polarization which effectively changes the interactions strength. The first diagram is shared by QED and QCD which renders the interaction stronger at shorter distance (screening). The second diagram arising from the nonlinear interaction between gluons in QCD has the antiscreening effect, which makes the coupling weaker at short distance.

In QCD, the same differential equation for the strong coupling constant holds. However, the β function is now different

$$\beta(\alpha) = -\frac{\beta_0}{2\pi}\alpha^2 + ... \tag{1.19}$$

where $\beta_0 = 11 - \frac{2}{3}n_f$ with n_f is the number of active quark flavor. The second term in β_0, $-\frac{2}{3}n_f$, comes from quark-antiquark pair effect in the first diagram in Fig. 1.4. It scales like the number of quark flavors and is negative (as it would be in QED). However, the first term, 11, has the opposite sign and comes from the non-linear gluon contribution shown in the second diagram (these are absent for QED). Thus the gluon self-coupling has an anti-screening effect! From the renormalization group equation, the coupling constant of QCD can be shown to have the following scale-dependence

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln(\mu/\Lambda_{\rm QCD})} \ , \tag{1.20}$$

which goes to 0 as the momentum scale $\mu \to \infty$ or the distance approaching 0! This strange behavior of the strong coupling has been verified in high-energy experiments to very high precision, as shown in Fig. 1.5, where Q stands for the running scale. The integration constant $\Lambda_{\rm QCD}$ is an intrinsic QCD scale, discussed below.

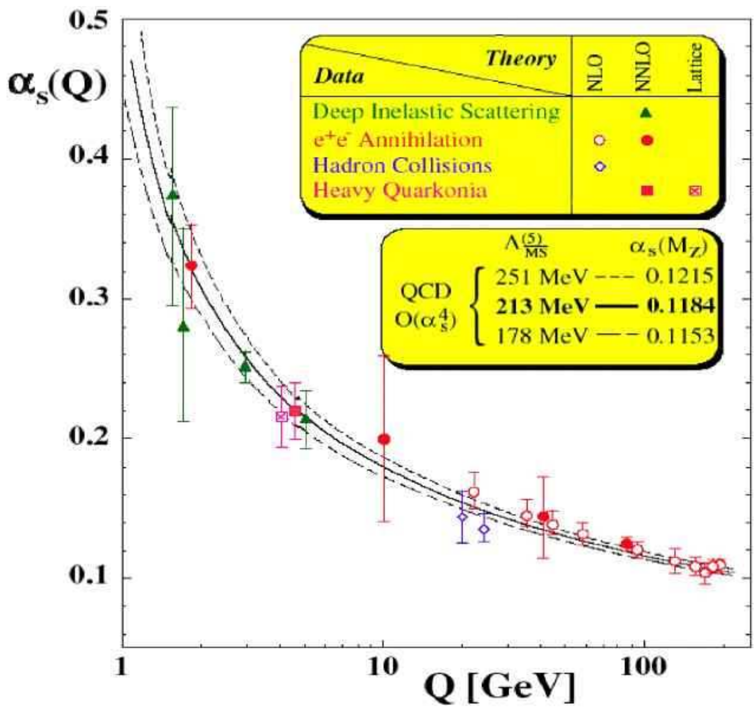

Figure 1.5: The running of the QCD coupling constant as a function of momentum transfer, experimental data vs. theoretical prediction.

Because of asymptotic freedom, the strong interaction physics can now be calculated in perturbation theory when the momentum transfer is large. In particular, the one-gluon exchange diagram (Fig. 1.3) becomes a good approximation for quark interaction as $q^2 \gg \Lambda^2_{\rm QCD}$. At present, the most accomplished result of QCD research is in the perturbative region, where many experimental data have been explained well by perturbative QCD (pQCD). One must emphasize, however, the force between quarks does not get weaker at the shorter distance, despite the fact that the coupling does. In fact, the force still grows at short distance in an asymptotic free theory!

Strong Interaction Scale $\Lambda_{\rm QCD}$

The running coupling introduces a dimensional parameter $\Lambda_{\rm QCD}$, which sets the scale at which the coupling constant becomes large and the physics becomes nonperturbative. In fact, $\Lambda_{\rm QCD}$ simply sets the scale for strong interaction physics. In the $\overline{\rm MS}$-scheme with 3 quark flavors,

$$\Lambda_{\rm QCD} \sim 250 \text{ MeV} . \tag{1.21}$$

All dimensionful QCD results without external momenta scale with this parameter when quark masses are negligible. For instance, it is largely responsible for the mass scale of protons and neutrons, and hence the mass scale of the baryonic mass in the Universe. On the other hand, pure QCD predictions with massless quarks are independent of $\Lambda_{\rm QCD}$ and hence are pure numbers.

1.4 Chiral Symmetry and Its Spontaneous Breaking

In the last section, we have introduced the strong interaction scale $\Lambda_{\rm QCD}$ through the running QCD coupling α_s. With this, we can introduce the notion of light and heavy quarks. Briefly speaking, light quarks are ones having masses much smaller than $\Lambda_{\rm QCD}$, and heavy quarks having masses much larger than that. Clearly, the up and down flavors are qualified for light quarks, whereas the charm, bottom, and top may be regarded as heavy. The strange quark is an exception, it appears neither light nor heavy, and is more difficult to deal with theoretically. In some cases, it can be regarded as light, in others it is closer to being heavy.

Here we focus on the light up and down quarks, which are perhaps the most relevant to the real world. To understand the physics of the light quarks, it is convenient to consider a theoretical limit in which their masses are exactly zero. We argue later that the physics of the real world is not so different from this theoretical limit.

Chiral Symmetry

If a quark has a zero mass, then the spin of the quark can either be in the direction of motion, in which case, we call it a *right-handed* quark, or in the opposite direction, in which case, we call it a *left-handed* quark. Since a massless quark travels at the speed of light, the handedness or chirality of the quark is independent of any Lorentz frame from which the observation is made.

The chirality can be selected by the Dirac matrix γ_5 because the free hamiltonian commutes with it. We then define a projection operator $P_\pm \equiv (1 \pm \gamma_5)/2$ to project out left and right-handed (chirality) quarks,

$$\begin{aligned} \psi_{Lf} &= \frac{1}{2}(1-\gamma_5)\psi_f , \\ \psi_{Rf} &= \frac{1}{2}(1+\gamma_5)\psi_f . \end{aligned} \tag{1.22}$$

where f labels different flavors. The total quark field is simply a linear combination of the two,

$$\psi = \psi_L + \psi_R . \tag{1.23}$$

If one plugs the above decomposition into the QCD lagrangian without the mass term, the quark part splits into two terms

$$\mathcal{L}_q = \mathcal{L}_q(\psi_L) + \mathcal{L}_q(\psi_R) , \tag{1.24}$$

with each depending on either the left-handed, or the right-handed field, but not both. In other words, the QCD interaction does not couple the left and right-handed quarks which appear to live in separate worlds.

Because the up and down quarks are both massless, they may be regarded as two independent states of the same object forming a two component spinor in "isospin space," in the same way as the $\pm 1/2$ projection states of a spin-1/2 object. The lagrangian density is symmetric under the independent rotations of ψ_L and ψ_R in the left and right isospaces. More explicitly, under

$$\begin{pmatrix} u'_{L,R} \\ d'_{L,R} \end{pmatrix} = U_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \tag{1.25}$$

where $U_{L,R}$ are 2×2 unitary matrices, the quark part of the lagrangian is invariant. We then say the QCD lagrangian has a chiral symmetry $U(2)_L \times U(2)_R$.

Since a 2 unitary matrix can be decomposed into product of a phase and a special unitary matrix with unit determinant, the chiral symmetry can be decomposed into

$$U(2)_L \times U(2)_R = SU(2)_L \times SU(2)_R \times U_L(1) \times U_R(1) \ . \tag{1.26}$$

The symmetry $SU(2)_L \times SU(2)_R$ has not been seen explicitly in particle spectrum or scattering matrix element, in the same way as the rotational symmetry in 3-dimensional motion of a quantum particle. Rather, it is a "hidden" symmetry, in the sense that it breaks *spontaneously* into the so-called *isospin* subgroup $SU(2)_{L+R}$ which corresponds to rotations $U_L = U_R$. Isospin symmetry is manifest in particle spectrum and scattering, and will be discussed further in Chapter 3.

A simple example of spontaneous symmetry breaking

Spontaneous symmetry breaking is a common phenomenon in physics: it happens in ordinary quantum mechanics and in many cases in condensed matter physics such as the spontaneous magnetization of a ferromagnet. A simple quantum mechanical example is shown here to illustrate what is going on. Consider a particle moving in a one-dimensional symmetric potential shown in Fig.1.6. The hamiltonian is symmetric under $x \to -x$, or parity transformation. Because of this symmetry, the wave functions of the eigenstates can be taken to be either symmetric and antisymmetric under parity. In particular, the ground state of the wave function $|0\rangle$ is always symmetric,

$$P|0\rangle = |0\rangle \ , \tag{1.27}$$

because the kinetic energy is minimized this way.

As the barrier in the middle gets higher, an interesting feature in the spectrum develops: the energy levels become clustered as doublets. The wave function of the lower level in each doublet is symmetric and that of the upper one is antisymmetric. The difference of the energy in the doublet is exponentially small as a function of the barrier height. In fact, as the barrier in the middle goes to infinity, all energy levels are exactly two-fold degenerate, including the ground state. Physically, this must be the case, because we effectively have two identical infinite square wells.

Thus, there are an infinite number of ways to choose the ground wave function when the degeneracy arises. However, some of the choices are the most natural, i.e., we take as the ground state when the particle is either in the left well, or in the right well. Placing a particle in both wells simultaneously is theoretically possible, but hard to realize in practice. Although the two

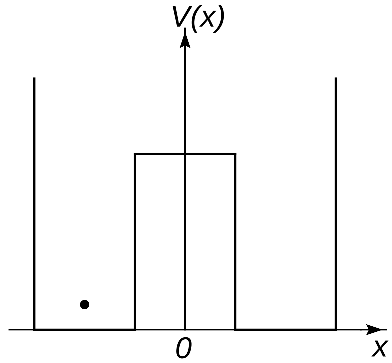

Figure 1.6: A quantum mechanical particle moving in a symmetric potential well with a finite barrier in the middle. The ground state wave function is left-right symmetric. However, when the barrier height approaches infinity, the ground state becomes doubly degenerate and a natural choice for the ground state wave function no longer has left-right symmetry.

choices are equally natural, once a choice is made, the ground state is unique. The other state is entirely decoupled because no operator in the Hilbert space can create a transition to that from any states built from the ground state (super-selection rule). If one accepts this, the ground state wave function no longer have a left-right symmetry under $x \to -x$. The symmetry is said to have been spontaneously broken.

A well-known example of spontaneous symmetry breaking is the magnetization of a magnetic material below a critical temperature T_c. Above T_c, the material has no net magnetic moment and is rotationally symmetric as the underlying hamiltonian. Below T_c, it develops a spontaneous magnetization which breaks the SO(3) rotational symmetry. All possible magnetization directions are physically equivalent, corresponding to different degenerate vacua. Once a direction is chosen, the system is still rotationally-invariant around the magnetization direction, or equivalently, symmetric under the subgroup SO(2).

Spontaneous Breaking of Chiral Symmetry and Pion as a Goldstone boson

The chiral symmetry is a continuous symmetry in the sense that it contains infinitely many transformations which are smoothly connected to one another. In the case of spontaneous breaking of a continuous symmetry, there are usual infinitely many degenerate vacua, which give rise to massless excitations referred to as Goldsone bosons. In the case of magnetization, the Goldstone bosons is the massless spin wave.

A simple way to understand the chiral symmetry breaking is to consider the dynamics of the scalar fields ϕ_i

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - V(\phi) \ . \tag{1.28}$$

If $V(\phi)$ is invariant under 4-dimensional rotation $\phi_i \to L_{ij}\phi_j$, the system has an $O(4) \sim SO(3) \times SO(3)$ symmetry. Taking as a simple assumption,

$$V(\phi) = -\frac{1}{2}\mu^2\phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2 \tag{1.29}$$

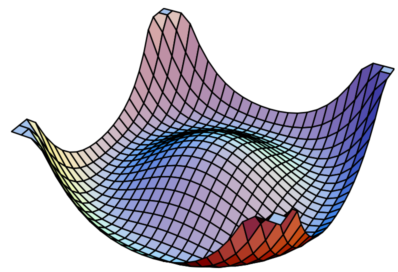

Figure 1.7: The scalar field potential when spontaneous symmetry breaking occurs. The minimum of the potential is now a hypersurface, which is not invariant under SO(4) rotation.

The SO(4) symmetry is spontaneously broken to SO(3) when $\mu^2 \leq 0$. Indeed as shown in Fig. 1.7, the minimal of the potential corresponds to

$$\phi_i^2 = c^2 = \sqrt{-\lambda/\mu^2} \ , \tag{1.30}$$

which has infinite number of solutions, corresponding to an infinite number of vacua, all obtained from each other by some O(4) rotations. The physical vacuum can be taken to be one of the infinite possible solutions. Once a solution is chosen, for example, $\phi_i = (0, 0, 0, c)$, this vacuum is still invariant under the SO(3) rotations of the first three dimensions. This corresponds to breaking of symmetry from SO(4) to SO(3), analogue of the isospin group.

Imagine we start with a common vacuum at every point of space and time, and make a rotation differently at different points of spacetime. The energy of the new state must be proportional to the derivative of the rotations in space, because a common rotation generates no energy. As the derivative vanishes, the energy must go to zero as well. Therefore, the excitation associated with the rotation has no mass. This is the Goldstone boson! In case of a magnetic, the Goldstone boson corresponds to massless spin waves.

Since in the case of the chiral symmetry, there are three independent ways to make rotations away from a chosen vacuum, just like the scalar field discussed above, we have three possible Goldstone bosons. These have been identified as negative parity pions $\pi^{\pm,0}(x)$, first observed in 1950's. The fact that chiral symmetry is not exact ($m_u \neq m_d \neq 0$, see Table 1.1) leads to a small (relative to $\Lambda_{\rm QCD}$) pion mass $m_\pi \sim 140$ MeV. Because of the low-energy scale involved, the dynamics of the pions can be described by an effective theory.

1.5 Color Confinement

One of the prominent features of QCD at low-energy is color-confinement: *Any strongly interaction system at zero temperature and density must be a color singlet at distance scale larger than* $1/\Lambda_{\rm QCD}$. As a consequence, isolated free quarks cannot exist in nature (quark confinement). The color confinement of QCD is a theoretical conjecture consistent with experimental facts. To prove it in QCD is still a challenge that has not been met.

Suppose, for example, we have a quark-antiquark pair which is in a color singlet state. One may try to separate the quark from the antiquark by pulling them apart. The interaction between the quarks gets stronger as the distance between them gets larger, similar to what happens in a spring. In fact, when a spring is stretched beyond the elastic limit, it breaks to produce two springs. In the case of the quark pair, a new quark-antiquark pair will be created when pulled beyond certain distance. Part of the stretching energy goes into the creation of the new pair, and as a consequence, one cannot have quarks as free particles.

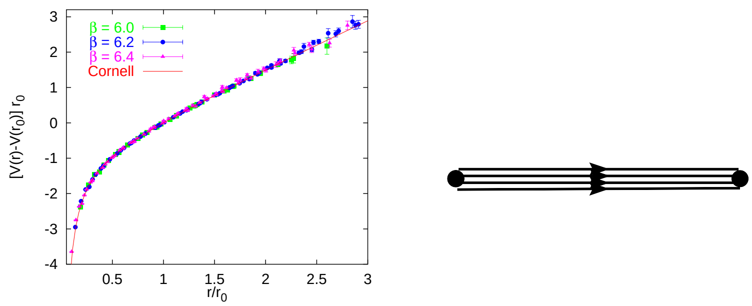

Figure 1.8: Left: potential between a pair of heavy quark-antiquark when there are no light quarks in the QCD vacuum. Right: the gluon flux tube between a pair of quark-antiquark produced by the so-called dual Meissner effect. The flux tube is believed to be responsible for the linear confinement potential.

The above discussion is in some sense a sort of self-consistent speculation. To understand really what happens, one must make calculations in QCD at large distance scales where, according to the renormalization group equation, the coupling becomes very strong. Such calculation is very difficult at present time. The only way we know how to solve QCD in the strong coupling regime is to simulate the theory on a finite spacetime lattice, or in short *lattice QCD*. One way to find the potential energy when the quarks are pull apart is to calculate quark-antiquark potential in a world where there is no quark-antiquark pair creation (quenched QCD). QCD simulations on lattice shows that the quark potential does increase linearly beyond the distance of a few fermis, as shown on the left panel in Fig. 1.8.

The origin of the linear potential between quarks may be traced to the so-called flux tube: a string of gluon energy density between the quark pair, shown on the right panel in Fig. 1.8. In QED, the electric lines between positive and negative charges spread all over the space, generating a $1/r$ potential. In QCD, however, the vacuum acts like a dual superconductor which squeezes the color electric field to a minimal geometrical configuration—a narrow tube. [In a normal superconductor, the magnetic flux is expelled from the interior of the metal, as shown in the Meissner effec.] It costs energy for the flux to spread out in space. The tube roughly has a constant cross section and with constant energy density. Because of this, the energy stored in the flux increases linearly with the length of the flux. The flux tube has been seen in numerical simulations. If one calculates the gluon energy in lattice QCD, one can see the flux tube from the the plot of energy density in space.

1.6 Hadrons and Nuclei

It is rather odd that in this introductory chapter, we spent so much space to talk about the something we don't usually "see" in low-energy experiments, quarks and gluons. What we usually observed in experimental apparatus are hadrons and nuclei which are bound states of these basic building blocks.

As we will discussed in the future chapters, a meson is made of the quantum number of a quark and anti-quark pair. For example, the pion has the quantum number of $\bar{u}u - \bar{d}d$, $\bar{u}d$, and $\bar{d}u$, naturally color singlets. On the other hand, a baryon is made from the quantum numbers of 3 quarks, which can form a color singlet because the SU(3) group multiplication rule says

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} \tag{1.31}$$

where **1** represents color singlet. For example, a proton is made of two up quarks and one down quark. There are other baryons such as Λ, Ω and the excited states of these particles as well. Baryons and mesons together are called *hadrons*, the bounds states of strong interactions.

The lowest mass baryons are neutrons and protons, which are together called *nucleons*. Protons are stable, whereas neutrons have a lifetime about 10 min. There are attractive interactions between protons and neutrons, which are the residual color interactions, just like the Van der Waals forces between neutral atoms and molecules. These nuclear forces are responsible for binding the nucleons together to form atomic nuclei, and the origin of atomic energy.

We will discuss extensively in this course the structures and interactions of hadrons and nuclei.

1.7 Electroweak Interactions

Weak interactions are mediated through W and Z vector bosons which play a similar role to photons and gluons in QED and QCD, respectively. In particular, Z boson is flavor-conserving and charge neutral, and therefore is very much like the photon. However, these weak gauge bosons are quite massive, 80.4 GeV/c^2 for W and 91.2 GeV/c^{2}for Z, and mediate a force that acts on a very short range with an extremely weak strength. The W bosons are charged and do change flavor. For instance, the emission of a W^+ changes an up quark into a down quark. It also changes an electron into an electron neutrino, a muon into a muon neutrino.

The electromagnetic and strong interactions involve just the vector current $j^\mu = V^\mu = \psi\gamma^\mu\psi$ and hence conserve parity. However, as was discovered in 1957 by Lee, Yang, and Wu, the weak interaction involves parity violation, and the axial vector current $A^\mu = j_5^\mu = \psi\gamma^\mu\gamma_5\psi$ also participates. Under a parity transformation $\vec{r} \to -\vec{r}$, the vector current transforms as $V^\mu \to -V^\mu$ whereas $A^\mu \to A^\mu$. Analogous to vector currents, we can define nine axial currents in terms of the flavor structure of the light up, down and strange quarks,

$$A^{\mu a} = \overline{\psi}\gamma^\mu\gamma_5\frac{\lambda^a}{2}\psi \ , \tag{1.32}$$

which will be useful later. The singlet axial current ($a = 0$) is not really conserved due to the so-called $U_A(1)$ anomaly.

A more formal discussion of weak interactions starts with definition of weak charges. The weak charge assignment is related to the chiral properties of the quarks introduced in a previous section.

The standard model assumes the left-handed up and down quarks form a two-component vector,

$$q_L^1 = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{1.33}$$

transforming under $SU(2)_L$ (weak isospin) as a doublet, whereas u_R and d_R transform as singlets. The superscript 1 here refers to the first generation. The pattern repeats for charm and strange quarks, the second generation, top and bottom the third. In addition, the quarks carry $U_Y(1)$ weak (scalar) hypercharge Y, defined through

$$Q = I_3 + \frac{Y}{2} \ , \tag{1.34}$$

where $I_3 = T_3$ is the third component of the weak $SU(2)_L$ isospin and Q the electric charge. The free quark and lepton lagrangian is invariant under the above weak isospin and hypercharge transformation.

To make the $SU(2)_L \times U_Y(1)$ symmetry local, we introduce gauge fields W_i^μ for $SU(2)_L$ and B^μ for $U_Y(1)$. The covariant derivative for the weak doublet is

$$D^\mu = \partial^\mu + ig\frac{\vec{\tau}}{2} \cdot \vec{W}^\mu + ig'\frac{Y}{2} \cdot B^\mu \ . \tag{1.35}$$

After spontaneous breaking of $SU(2)_L \times U_Y(1)$ through a Higgs doublet $H = (\phi^+, \phi^0)$, only $U_{\rm em}(1)$ remains. The charge neutral gauge bosons W^0 and B will mix to give massless photon field A^μ and massive weak gauge boson field Z^μ

$$Z^\mu = \frac{gW_3^\mu - g'B^\mu}{\sqrt{g^2 + g'^2}} = \cos\theta_W W_3^\mu - \sin\theta_W B^\mu \tag{1.36}$$

$$A^\mu = \frac{g'W_3^\mu + gB^\mu}{\sqrt{g^2 + g'^2}} = \sin\theta_W W_3^\mu + \cos\theta_W B^\mu \ , \tag{1.37}$$

where we have introduced the Weinberg mixing angle θ_W. Since the photon field A^μ mediates the electromagnetic interaction, the couplings are related by

$$e = g\sin\theta_W = g'\cos\theta_W \ . \tag{1.38}$$

With the above definition, the coupling of Z-boson to quarks and leptons is

$$\mathcal{L}_{\rm int}^Z = -\frac{g}{\cos\theta_W}\overline{\psi}_i\gamma^\mu(g_V^i - g_A^i\gamma^5)\psi_i Z_\mu \tag{1.39}$$

where the vector and axial coupling constants are $g_V^i = I_3 - 2q_i\sin^2\theta_W$ and $V_A^i = I_3$, respectively. Experimentally (in the $\overline{\rm MS}$-scheme and at the scale M_Z) the measured value is $\sin^2\theta_W = 0.23120(15)$.

In the standard model with 3 light quarks, the W-boson interaction current is

$$J_W^\mu/2 = \cos\theta_C\bar{u}_L\gamma^\mu d_L + \sin\theta_C\bar{u}_L\gamma^\mu s_L + ... \ , \tag{1.40}$$

where $\theta_C = 15^\circ$ is called the Cabibbo angle. The interaction violates the parity maximally, it is a $V - A$ from, $\bar{\psi}\gamma^\mu(1 - \gamma_5)\psi$. Therefore, the quarks of the first generation can be converted to those of the second generation through W-bosons! This is called flavor mixing. In fact, the most general flavor mixing in the standard model with six quarks goes as

$$J_W^\mu/2 = \bar{u}_L\gamma^\mu d'_L + \bar{c}_L\gamma^\mu s'_L + \bar{t}_L\gamma^\mu b'_L \ , \tag{1.41}$$

where the primed quarks (flavor eigenstates) are

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{1.42}$$

where the 3×3 matrix is a unitary matrix (Cabbibo-Kobayashi-Maskawa mixing matrix) satisfying the unitarity condition, for example,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \ . \tag{1.43}$$

The unitarity matrix can be parameterized in terms of three mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ and a phase δ, four independent parameters. The standard parametrization is

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{1.44}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. An alternative parametrization exhibiting the heirachical structure of the mixing element is

$$V = \begin{pmatrix} 1-\lambda^2/ & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{1.45}$$

which was advocated by Wolfenstein.

The interaction between the W boson and the charge-changing current is

$$\mathcal{L} = -\frac{g}{2\sqrt{2}}W^{+\mu}J_{\mu W} + \text{h.c.} \tag{1.46}$$

If one integrates out the W-boson, one obtain the effective weak interaction in a form which was written down by Fermi over 50 years ago,

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}}J_W^\mu J_{W\mu} \tag{1.47}$$

where $G_F = 1.16637\times 10^{-5}$ GeV^{-2} is called Fermi-decay constant. The relation to the Fermi-decay constant G_F and the weak coupling g is, $g^2/8m_W^2 = G_F/\sqrt{2}$.

1.8 ***Background Information for Chapter 1***

1.8.1 SU(3) group and Gell-Mann matrices

Consider a 3-dimensional *complex* vector space, in which a vector ψ is represented by a column matrix of 3 complex numbers,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} . \tag{1.48}$$

A "rotation" in this space can be accomplished by a 3×3 unitary matrix U,

$$\psi'_i = U_i{}^j \psi_j \tag{1.49}$$

where the unitary condition $U^\dagger U = UU^\dagger = 1$ guarantees that the new vector ψ' has the same norm, defined as $\sum_i \psi_i^* \psi_i$, as the original one.

All 3×3 unitary matrices form a group $U(3)$ under the matrix multiplication. The determinant of U has a unit norm, $|\mathrm{Det}U| = 1$, and hence $\mathrm{Det}U = e^{i\theta}$. The subset of $U(3)$ rotations with $\mathrm{Det}U = 1$ form a subgroup, called $SU(3)$. The conditions $U^\dagger U = UU^\dagger = 1$ and $\mathrm{Det}U = 1$ leave U depending on 8 real parameters, which we will call θ^a ($a = 1, 2, ..., 8$). We can choose these parameters such that when $\theta^a = 0$, U reduces to the unit matrix I. When θ^a are small, we can expand U at $\theta^a = 0$ to get,

$$U = I + i \sum_{a=1}^{8} \theta^a t^a + ... \tag{1.50}$$

where t^a are 3×3 hermitian matrices, which are called the generators of the $SU(3)$ group, in the sense that the other group element may be constructed from these generators. Clearly, the choice of t^a depends on the choice of parameters.

A convenient choice of the SU(3) generators $t^a = \lambda^a/2$ by Gell-Mann is

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}; \quad \lambda_0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};
\end{aligned} \tag{1.51}$$

The generators satisfy the $SU(3)$ commutation relations ($SU(3)$ algebra),

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = i f^{abc} \frac{\lambda^c}{2} , \tag{1.52}$$

where f_{abc} is totally antisymmetric structure constants, the numerical values of which can be found in many textbooks (for example, Itzykson and Zuber).

The vector ψ_i forms a 3-dimensional fundamental representation of SU(3), and labelled by **3**. All the complex conjugate matrices U^* form another representation of $SU(3)$, which is called $\bar{\mathbf{3}}$. Clearly ψ_i^* space is a $\bar{\mathbf{3}}$.

Consider adding color charges of two quarks symbolically denoted as q^i and $\tilde{q}^i$. The easiest way to accomplish the addition is by the so-called tensor method. Multiplying the two 3-dimensional spaces together, we get a 9-dimensional space spanned by basis $q^i\tilde{q}^j$. The ij symmetric part,

$$\frac{1}{2}(q^i\tilde{q}^j + q^j\tilde{q}^i) \ , \tag{1.53}$$

forms a 6-dimensional subspace, **6**. The ij antisymmetry part,

$$\frac{1}{2}(q^i\tilde{q}^j - q^j\tilde{q}^i) \ , \tag{1.54}$$

form a 3-dimensional subspace. This, however, is not the same as the 3D color space of the original quark, because under an SU(3) transformation the basis states in this new 3D space transform like complex conjugate of the original quarks. For this reason, we have a new charge state $\bar{\mathbf{3}}$. To conclude, when we add two-triplets of color charges, we get a **6** and a $\bar{\mathbf{3}}$, or simply $\mathbf{3}\times\mathbf{3} = \mathbf{6}+\bar{\mathbf{3}}$.

What if one adds together the color charges of a quark and antiquark? The combination $q^1\bar{q}^1 + q^2\bar{q}^2 + q^3\bar{q}^3$, is invariant under SU(3) transformation and is a color singlet denoted by **1**. A color-singlet state is also color-natural, or white. The combinations $q^i\bar{q}^j$ with the color-neutral state subtracted form an 8-dimensional representation of the SU(3) group, **8**. Therefore, $\mathbf{3}\times\bar{\mathbf{3}} = \mathbf{1}+\mathbf{8}$.

More complicated multiplications of the $SU(3)$ representations can be obtained by the so-called tensor method, for example

$$\mathbf{3}\times\mathbf{3}\times\mathbf{3} = \mathbf{1}+\mathbf{8}+\mathbf{8}+\mathbf{10} \tag{1.55}$$

can be obtained this way.

1.8.2 Lagrangian Density, QED and Feynman Rules

The equations of motion for physical variables can be derived from the lagrange principle. In classical physics, a lagrangian can be constructed as the difference between kinetic energy T and potential energy V. The action is the time integral of the lagrangian,

$$S = \int dt L \ . \tag{1.56}$$

If a lagrangian depends on a coordinate q, the minimization of the action leads to the following Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \ , \tag{1.57}$$

where $\dot{q}$ is the time-derivative.

QED lagrangian density is constructed out of the electron Dirac field ψ and phone vector field A^μ ,

$$\mathcal{L} = \psi(i \not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{1.58}$$

where D is a covariant derivative $D^\mu = \partial^\mu + ieA^\mu$, and e is the electric charge. The Euler-Lagrange equations for the electron and photon fields are

$$(i \not{D} - m)\psi = 0 \quad (1.59)$$

$$\partial_\mu F^{\mu\nu} = eJ^\nu \ , \quad (1.60)$$

where the first one is the Dirac equation for electron in an electromagnetic field, and the second one is the Maxwell equation with the electron vector current $J^\mu = \bar{\psi}\gamma^\mu\psi$.

The Feynman rule for the electron propagator with four momentum k is

$$\frac{i}{\not{k} - m + i\epsilon} \ . \quad (1.61)$$

The photon propagator depends on the choice of gauge. Because not all degrees of freedom of A^μ are physical, additional constraint must be imposed to make the physical photon propagation finite. The most frequently used gauge is Feynman gauge in which the photon propagator is simply,

$$\frac{-ig^{\mu\nu}}{k^2 + i\epsilon} \ . \quad (1.62)$$

The electron photon interaction vertex can be read off directly from the above lagrangian density,

$$-ie\gamma^\mu \quad (1.63)$$

Using the above Feynman rules, one can calculate any QED processes.

1.9 Problem Set

1. Work out the product of $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ in term of irreducible representations of SU(3).
2. Why the antisymmetric part of $\mathbf{3} \times \mathbf{3}$ corresponds to a $\bar{\mathbf{3}}$?
3. Show that (1.6) is gauge-invariant and $F_{\mu\nu}$ transforms like $UF_{\mu\nu}U^\dagger$ under SU(3) gauge transformation.
4. Show that the color factor for two quarks in $\mathbf{6}$ is 1/3, and that in $\bar{\mathbf{3}}$ is $-2/3$.
5. Solve the renormalization group equation for the QCD coupling to get the formula in Eq. (20).
6. In the SO(4) model for spontaneous symmetry breaking, identify the Goldstone boson degrees of freedom as well as the group under which the physical vacuum is invariant.