Physics 880.05: Problem Set #4

The problems are due by 5pm on Friday, December 11. See the online lecture notes for details of any of the individual topics covered in this problem set. Please give feedback early and often (and stop by to ask about anything).

1. **Spin-1/2 fermions in a (1+1)–dimensional lattice: formal.** We saw in class (lecture 15) that the partition function for a system of spin 1/2 fermions with a 2-body interaction can be written as

$$\mathcal{Z} = \int \mathcal{D}\sigma \, \det[M^\dagger M] e^{-S_g[\sigma]} \tag{1}$$

   where $M = M[\sigma]$ is the fermion operator that results from using the Hubbard-Stratonovich trick (in any one of its forms). Use the lecture notes and results from the previous problem set to help answer the following.

   (a) Find $M[\sigma]$ and $S_g[\sigma]$ for the particular case of an attractive zero-range interaction, i.e., starting from the Euclidean action

$$S_E[\psi^\dagger, \psi] = \int dx d\tau \, \left[\psi^\dagger \Big(\frac{\partial}{\partial \tau} - \nabla^2/2m \Big) \psi - g(\psi^\dagger \psi)^2 \right] . \tag{2}$$

   (b) Write down $M[\sigma]$ and $S_g[\sigma]$ in discretized spacetime as in Problem 4 of PS#3 (you may assume that the spatial lattice spacing a_x is 1, and take the temporal lattice spacing to be $a_t = \Delta\tau a_x$, where $\Delta\tau$ is an input parameter).

   (c) Show that $P[\sigma] = \det[M^\dagger M] e^{-S_g[\sigma]}$ is a well-defined probability measure, i.e., it is positive and bounded from above. (Hint: What kind of matrix is $M^\dagger M$?)

2. **Spin-1/2 fermions in a (1+1)–dimensional lattice: numerical.** We continue with the (1+1)–dimensional problem but now use numerical methods. The Metropolis algorithm is implemented to sample configurations of the field $\sigma(x, \tau)$ according to $P[\sigma]$ in the MATLAB code available on the home page (one_d_fermions.zip).

   (a) Check the correctness of FermionAction2.m by using FermionDeterminantTest.m with $c = 0$ to

      i. verify explicitly the matrix M entries (print it out) for a small size;

      ii. check the determinant by calculating it independently from the eigenvalues (warning: this is somewhat tricky—keep in mind the boundary conditions).

(b) Set up the code (which one?) to make 1000 thermalization sweeps (updates of the whole lattice) and then sample 1000 configurations of σ, allowing for 250 decorrelation sweeps between each sample. For this purpose, take the spatial direction to have $N_x = 32$ lattice points, and set the time direction to have 10, 20, and 30 points, with $\Delta\tau = 0.125$. (You might want to explore different values!) What is output by the code?

(c) Compute the average particle number and energy for $g = 0.1$ and $g = 2$ (see extra notes and interact as needed with Joaquin by email or in person).

(d) Bonus: implement sampling with the Hybrid Monte Carlo algorithm.

3. **Effective action in one dimension.** The idea here is to go through our treatment of the large-N expansion of the effective action (which we didn't finish in class but which is in the notes for lectures 17 and 18) and to evaluate it in the Bose limit, but in one spatial dimension (or 1+1 spacetime).

(a) Derive the one-dimensional expression for $\mathcal{E}_{\rm LO}$ corresponding to the one on page 205 of the notes. How does this compare to the result you got in Problem 3 of PS#3? Explain about the (slight) difference.

(b) Derive expressions for $\mathcal{E}_{\rm NLO}$ and for $\Pi_0(q_0, q)$, corresponding to the ones on page 207 (taking into account how $\mathcal{E}_{\rm NLO}$ is related to $\Gamma_{\rm NLO}$). BONUS: Evaluate the integral in $\Pi_0(q_0, q)$.

(c) Take the Bose limit as on page 208 and derive the LO and NLO terms for the energy density of a dilute Bose gas with a repulsive interaction. You will need to use the dimensional regularization (DR) formula on page 207. What happened to the kinetic energy (explain the physics)?