# H133: 1094 Session 3 $\boxed{\text{KEY}}$

*Write your name and answers on this sheet and hand it in at the end.*
**After the indicated time, move on to the next activity, even if you are not finished!**

## 1. Q6: Warm-up Problems on Rules 1-3 [10 min.]

a. Complex variable warm up. Express the following in the form a + bi: (a) (6+3i)(1-i) (b) $|-3-2i|^2$

a) $6-6i+3i+3=\boxed{9-3i}$

b) $(-3-2i)\cdot(-3+2i) = 9-6i+6i-4i^2 = 9+4 = \boxed{13}$ [or $|a+bi|^2 = a^2+b^2$]

b. Rule 1: State Vector. The state vector |psi> that describes the state of a quanton at a given time should be *normalized.* That means that <psi | psi> = 1. Do Problem Q5B.10, part (b).

$$|\psi\rangle = \begin{bmatrix} a(1+i) \\ a i \end{bmatrix} \Rightarrow \langle\psi|\psi\rangle = \begin{bmatrix} a(1-i) & -ai \end{bmatrix}\begin{bmatrix} a(1+i) \\ ai \end{bmatrix} = 1 = a^2\cdot 2 + a^2 = 3a^2 \Rightarrow \boxed{a=\frac{1}{\sqrt{3}}}$$

(or $a = -\frac{1}{\sqrt{3}}$)

c. Rule 2: Eigenvectors. The different eigenvectors for an observable should be *orthogonal* to each other. E.g., <+x | -x> = 0. Show using Table Q6.1 that |+theta> and |-theta> are orthogonal.

From the table,

$$\langle +\theta| = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \end{bmatrix} \text{ and } |-\theta\rangle = \begin{bmatrix} i\sin\theta/2 \\ \cos\theta/2 \end{bmatrix} \Rightarrow \langle +\theta|-\theta\rangle = i\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = \boxed{0}$$

note →

d. Do Problem Q6T.3. Which rule is this an example of? [Hint: You shouldn't need to do a calculation.]

Applying Rule 3, Collapse, the outcome is $|-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{D.}$

## 2. Rule 4: Outcome Probability [12 min.]

Do the following problems by applying Rule 4, which says that the probability of an outcome a with eigenvector |A>, given initial state |psi0>, is: $P(a) = |\langle psi0 \mid A\rangle|^2$
When doing each problem below, identify |psi0>, |A>, and a, and calculate P(a).
Use Table Q6.1 to find the eigenvectors you'll need.

a. Do Problem Q6T.1. $|\psi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|A\rangle = |+x\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $a = +S$

$$\langle\psi_0|A\rangle = 1\cdot\tfrac{1}{\sqrt{2}} + 0\cdot 1/\sqrt{2} = \tfrac{1}{\sqrt{2}} \Rightarrow P(A) = |\langle\psi_0|A\rangle|^2 = \tfrac{1}{2} \Rightarrow \boxed{C.}$$

b. Do problem Q6T.2. (Note that it says *anti-aligned* with +x.)

$|\psi_0\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $A = |-x\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $a = -S$

$$\langle\psi_0|A\rangle = \tfrac{1}{2} - \tfrac{1}{2} = 0 \Rightarrow P(A) = 0 \Rightarrow \boxed{A.}$$

independent of $\theta$!

c. Do problem Q6B.1, part (a).

$|\psi_0\rangle = |+x\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $|A\rangle = |-\theta\rangle = \begin{bmatrix} i\sin\frac{1}{2}\theta \\ \cos\frac{1}{2}\theta \end{bmatrix}$ $\Rightarrow$ $\langle\psi_0|A\rangle = \frac{1}{\sqrt{2}}(i\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta)$

$a = -S$

$$|\langle\psi_0|A\rangle|^2 = \tfrac{1}{2}(\sin^2\tfrac{1}{2}\theta + \cos^2\tfrac{1}{2}\theta) = \boxed{1/2}$$

d. *If you still have time left,* check your answers with the PhET flash applet "Stern-Gerlach Experiment" (Start->Programs->PhET, choose "Quantum Phenomena" from the left menu, and click on the Stern-Gerlach icon). The simulation shows a source of atoms with controllable spin orientations and one to three Stern-Gerlach devices, whose orientation can be controlled. You can have an SGz device by setting the angle to zero, an SGx device by setting the angle to 90, or an SGtheta device with an intermediate angle. You also control which output channel is blocked. a) and b) work, but c) doesn't. In the simulation, $\theta$ is in the x-z plane, while in the book, $\theta$ is in the y-z plane, so different.

## 3. Rule 5: Superposition and Rule 6: Time-Evolution [12 min.]

Let's step through problem Q6S.3, using the Example at the top of page 106 as a guide. Use Table Q6.1 to find the eigenvectors you'll need.

a. We are told that the energy eigenstates are |+x>, with eigenvalue E0, and |-x>, with eigenvalue -E0. Write the initial spin state |psi(0)> at time t=0 as a superposition of |+x> and |-x> [i.e., the analog of equation (Q6.8)].

$$|\psi(0)\rangle = \begin{bmatrix}1\\0\end{bmatrix} = c_1\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} + c_2\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} \Rightarrow \begin{matrix}\frac{1}{\sqrt{2}}(c_1+c_2)=1\\ \frac{1}{\sqrt{2}}(c_1-c_2)=0\end{matrix} \Rightarrow \boxed{c_1=c_2=\frac{1}{\sqrt{2}}}$$

b. Now apply Rule 6 to find the spin state at time t, |psi(t)>. [If you have time, try to simplify your result.]

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_0t/\hbar}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} + \frac{1}{\sqrt{2}}e^{+iE_0t/\hbar}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^{-iE_0t/\hbar}+e^{+iE_0t/\hbar}\\ e^{-iE_0t/\hbar}-e^{+iE_0t/\hbar}\end{bmatrix}$$

[or use $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ right away] using (Q5.9c) and (Q5.9d) $= \begin{bmatrix}\cos(E_0t/\hbar)\\ -i\sin(E_0t/\hbar)\end{bmatrix}$

## 4. Q6.3: The Wavefunction [15 min.]

Start up the PhET applet "Quantum Bound States". This applet shows wavefunctions for the spatial subset of observables, as described in Q6.3.

a. Change the Potential Well using the pulldown menu on the upper right to Harmonic Oscillator. This has the potential energy $k*x^2/2$, which is the same as a spring. The possible energy eigenvalues are shown as horizontal lines (with energy in eV) while the corresponding eigenvectors (the wave functions) are shown below. Switch the Display (middle right) from Probability Density to Wave Function. Move the mouse from the lowest to the highest energy lines; you'll see the corresponding wave function (at t=0) in yellow below. List two ways in which the wave functions are analogous to standing waves on a string. • There are "fixed" boundary conditions at the ends (waves go to zero).
• With each higher energy/frequency, there is one additional node.
• The effective wavelength (spacing of crests) decreases with increasing energy/frequency.

b. Now switch back to Probability Density. The total area under a Probability Density curve is one, corresponding to the total probability of one for finding the particle somewhere. The area under the curve between two positions is the probability to find the particle in that region. Click on a few different energy lines to select some different states. For state E2, near what Positions (*give numbers*) are you most likely to find the particle? Where are you least likely to find it?

most likely: $\simeq \pm 0.6$ nm $\Rightarrow$ when $|\psi(x)|^2$ is largest
least likely: $\simeq \pm 0.2$ nm $\Rightarrow$ when $|\psi(x)|^2 \simeq 0$.

c. Quantitative estimation. Do problem Q6T.7, using Example Q6.2 as a guide. Explain your answer.

Count boxes for $|\psi(x)|^2$.
$1+1+1+4+4+1=12$ total. $4+1=5$ for $x>1$ nm $\Rightarrow$ Prob. $= \frac{5}{12} \Rightarrow$ (E)

d. Time dependence again. According to Rule 6, what is the time-dependent wave function |psi(t)> for a state composed only of eigenvector |E3>? What is the real part of this wave function? Find the energy of the state with E3 from the simulation and predict the period of the wavefunction. Then click on the state E3 (its line should turn red), display the real part of the wave function, and (roughly) measure the period using the clock in the lower left. Compare your prediction and measurement.

$\langle x|\psi(t)\rangle = e^{-iE_3t/\hbar}\langle x|E_3\rangle$ eigenvector $\quad \mathrm{Re}\langle x|\psi(t)\rangle = \cos(E_3t/\hbar)\langle x|E_3\rangle$ assuming this is real

period T defined by $\frac{E_3T}{\hbar} = 2\pi \Rightarrow T = \frac{2\pi\hbar}{E_3} = \frac{h}{E_3} = \frac{4.136\times10^{-15}\,\text{eV}\cdot\text{s}}{2.3\,\text{eV}} \doteq 1.8\times10^{-15}\,\text{s} = \boxed{1.8\,\text{fs}}$

Counting 10 periods, $10T = 18.1$ fs $\Rightarrow$ checks ✓