Chapter 6

Special Note: Professor Heinz may start lecturing on this chapter. He may not necessarily follow these notes. I will finish this chapter.

- We have explored the some of the evidence that led to the formulation of quantum mechanics. Now we need to start to develop the mathematic model for quantum mechanics which will allow us to explain the experiments we have discussed.
- We are going to take the following approach
  - Develop a set of rules for quantum mechanics. We will accept these rules as correct
  - Define something called a "Wavefunction". This is the key function (discrete and continuous) that define the state of our system.
  - We will see how the rules and the wavefunction allow us to predict our observations with SG devices and with the two-slit experiment with particles.

- Note: Quantum Mechanics is not like the other theories that we have studied in the past. In quantum mechanics the mathematical model makes a direct connection to the observation. We don't have a conceptual models or examples to help form a intuition about what is happening. That's just something we will have to live with in the strange world of quantum mechanics.

“Observables”

- All quanta have a set of *intrinsic characteristics* that help define what that object is
  - Mass, Electric charge, Spin, etc.
  - These never change for a particle and it is these quantities that define the particle.
    - An electron is an object with mass 0.511 MeV/c^2, -1 electric charge, and spin of ½ .

- All quanta also have a set of quantities which we call “observables”, which we can measure experimentally.
  - Position, Momentum, Energy, Spin Projection
  - We will deal mostly with these…but there are others too.
  - When we perform and experiment these are the quantities we normally determine and by measuring these we can often extract the *intrinsic* characteristics.

- We will divide “observables” into two different groups
  - Spin Subset (S_x, S_y, S_z)
  - Position Subset (x, p)
  - Included in these subsets are other variables that can be determined from the basic components of the subset.
    - e.g. $E = p_x^2/2m + V(x)$ can be determined from x, p.
  - Normally, these two subsets are independent of one another and therefore we can consider one at a time.

Rule #1
The State Vector Rule

- "The state of a quanton at a given time is described by using a normalized state vector |y> having a certain number of complex components. In the context of a certain subset of observables, a quanton's state vector has as many components as there are possible values for any one of the subset's basic observables. Even so, the components of this vector *do not* correspond to the values of that or any other observable."
- There is a lot going on here let's break it down
- (A) Components:
  - Complex…just a statement we use complex numbers.
  - Number is equal to the number of possible values (states)
    - e.g. Spin Projection S_z can have only 2 components (+s) and (-s).

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Component associated with +s

Component associated with -s

- (B) Vector is normalized

$$\langle\psi|\psi\rangle = \begin{bmatrix} \psi_1^* \\ \psi_2^* \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = |\psi_1|^2 + |\psi_2|^2 = 1$$

- (C) The values of the complex components are not the *values* of the observables…as we will see in a moment, they are related to the prob. of observing that comp.

This is in contrast to Newtonian Mechanics where the actual values of the observables (**x**,**p**) that define the state of the system.

Rule #2
The Eigenvector Rule

- "For every possible numerical value that an observable might have, there is an associated normalized state vector, which we call that value's **eigenvector**. (Conversely, the value corresponding to a given eigenvector is called that vector's **eigenvalue.**)"

Breakdown:

- This rule is describing a special set of vectors, one for each possible value an observable might have.
  - S_z (spin along z-axis) can have two possible values +s and –s. The two eigenvectors associated with these are

$$|+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , |-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  - S_x (spin along x-axis) can have two possible values +s and –s. The two eigenvectors associated with these are

$$|+x\rangle = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix} \quad , |-x\rangle = \begin{bmatrix} \sqrt{1/2} \\ -\sqrt{1/2} \end{bmatrix}$$

  - For S_y

$$|+y\rangle = \begin{bmatrix} \sqrt{1/2} \\ i\sqrt{1/2} \end{bmatrix} \quad , |-y\rangle = \begin{bmatrix} i\sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix}$$

Note: The eigenvectors for an observable are orthogonal:

$$\langle +x | -x \rangle = 0$$

- These vectors have a rather strange structure which we will not show where it comes from…some things need to wait for a more advanced course in QM. Notice that x, y, and z are not the same…but they are still arbitrary!

$$|+\theta_{zy}\rangle = \begin{bmatrix} \cos\frac{1}{2}\theta \\ i\sin\frac{1}{2}\theta \end{bmatrix} \quad , |-\theta_{zy}\rangle = \begin{bmatrix} i\sin\frac{1}{2}\theta \\ \cos\frac{1}{2}\theta \end{bmatrix}$$

Rule #3
The Collapse Rule

- "When we perform any experiment to determine the value of one of a quanton's observables, the experiment *determines* the observable's value by "collapsing" the quanton's state to a randomly selected one of that observable's eigenvectors, and yields the observable value corresponding to that eigenvector."

Breakdown:

- This is a very strange concept…one which gives many physicists difficulty…even Einstein struggled with this prompting him to say:
  - "God does not play dice with the universe"
- This rule implies that, in general, the quanton has no definite value of observable A **until** we try to measure it. Then it assumes one of it's possible values, a_n, and the quanton's state vector suddenly becomes the eigenvector, A_n, for that value, a_n .
- Example:
  - The spin state of an electron is

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

  - When then measure the S_z with an SG_z device.
  - If the electron emerges from the – terminal the state vector will be

$$|\psi\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

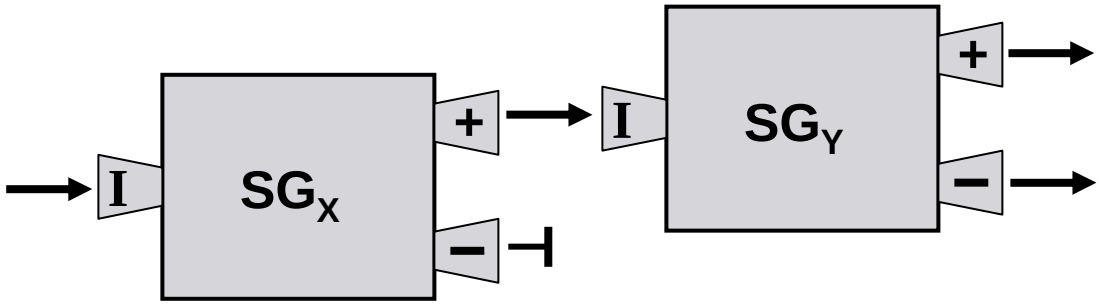
Rule #4
The Outcome Prob. Rule

- "In an experiment that determines the value of an observable, the probability of any given result (i.e., the probability that the quanton's state will collapse to that result's eigenvector) is the *absolute square of the inner product* of the quanton's original state and the result's eigenvector."

$$P(a) = \left|\langle \psi_o \,|\, A \rangle\right|^2$$

Breakdown:

- To understand what this is saying, let's do an example



- The state after the electron emerges from the SG_x device is (eigenvector for +x state)

$$|+x\rangle = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix}$$

- The probability that the electron will emerge from the – terminal of the SG_Y device is given by the inner product squared:

$$\left|\langle +x \,|\, -y \rangle\right|^2 = \left| \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix} \cdot \begin{bmatrix} i\sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix} \right|^2 = \left| \tfrac{1}{2} i + \tfrac{1}{2} \right|^2 = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2} \, (50\%)$$

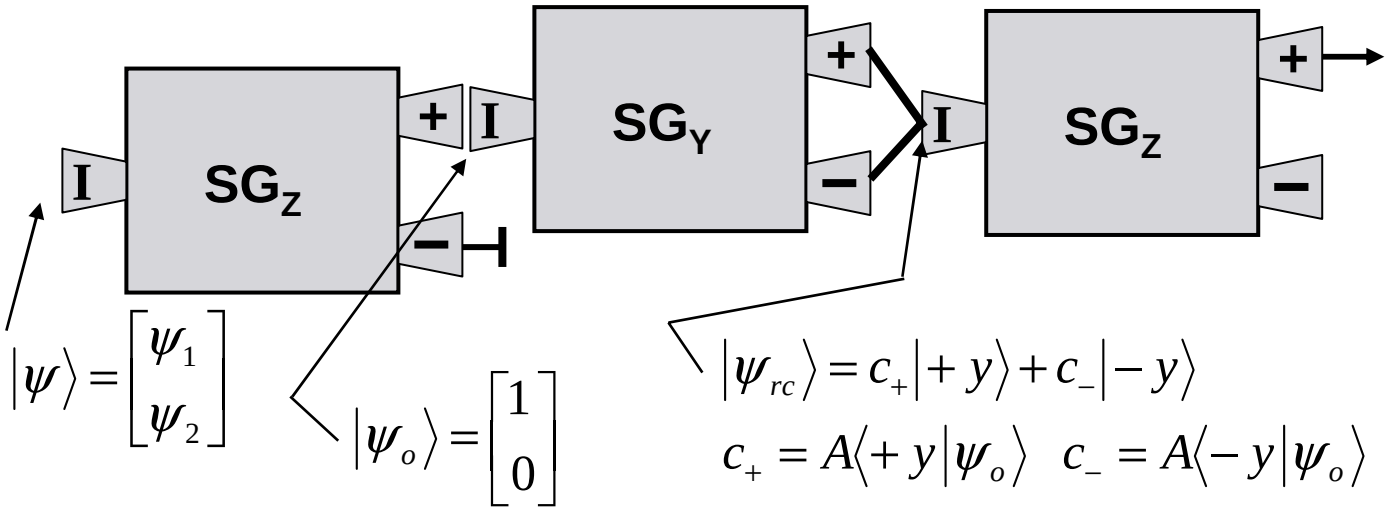
Rule #5
The Superposition Rule

- "Consider an experiment in which we take quantons in an initial state $|\psi_o\rangle$ and determine some observable whose possible values are *a,b,c*…and whose corresponding eigenvectors are |a>, |b>, |c>… If we take the subset of quantons determined to have values *a* or *b* and recombine them so that it is *completely impossible* to tell which value a given quanton was determined to have, then the state $|\psi_{rc}\rangle$ of the recombined quantons is the superposition

  $|\psi_{rc}\rangle = c_a|a\rangle + c_b|b\rangle$ where $c_a = A\langle a|\psi_o\rangle$, $c_b = A\langle b|\psi_o\rangle$

  and A is chosen to normalize $|\psi_{rc}\rangle$"

- To see how this works let's work with an example that we found very surprising:



Rule #5 (cont.)

- Now let's evaluate the coefficients c_+ and c_-:

$$c_+ = A\langle +y|\psi_0\rangle = A\begin{bmatrix}\sqrt{\frac{1}{2}}\\ i\sqrt{\frac{1}{2}}\end{bmatrix}^* \cdot \begin{bmatrix}1\\0\end{bmatrix} = A\sqrt{\tfrac{1}{2}}$$

$$c_- = A\langle -y|\psi_0\rangle = A\begin{bmatrix}i\sqrt{\frac{1}{2}}\\ \sqrt{\frac{1}{2}}\end{bmatrix}^* \cdot \begin{bmatrix}1\\0\end{bmatrix} = -iA\sqrt{\tfrac{1}{2}}$$

- Now write down what the full state vector is:

$$|\psi_{rc}\rangle = A\sqrt{\tfrac{1}{2}}\begin{bmatrix}\sqrt{\frac{1}{2}}\\ i\sqrt{\frac{1}{2}}\end{bmatrix} - iA\sqrt{\tfrac{1}{2}}\begin{bmatrix}i\sqrt{\frac{1}{2}}\\ \sqrt{\frac{1}{2}}\end{bmatrix} = \begin{bmatrix}\frac{A}{2}\\ i\frac{A}{2}\end{bmatrix} + \begin{bmatrix}\frac{A}{2}\\ -i\frac{A}{2}\end{bmatrix} = \begin{bmatrix}A\\0\end{bmatrix}$$

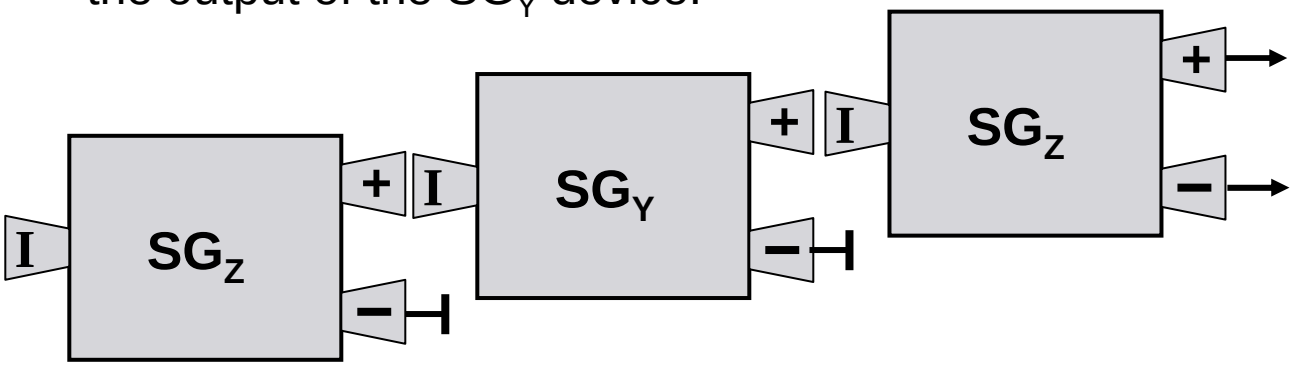
- For the vector to be normalized, for this case A=1.
- Now we see why 100% of the electrons come out of +z terminal of the third SG device, because the state vector coming ***into*** the final SG_Z device is the eigenvector for I+z>.

$$\text{Prob +z} = \left|\langle\psi_{rc}|+z\rangle\right|^2 = \left|\begin{bmatrix}1\\0\end{bmatrix}^* \cdot \begin{bmatrix}1\\0\end{bmatrix}\right|^2 = 1$$

$$\text{Prob -z} = \left|\langle\psi_{rc}|-z\rangle\right|^2 = \left|\begin{bmatrix}1\\0\end{bmatrix}^* \cdot \begin{bmatrix}0\\1\end{bmatrix}\right|^2 = 0$$

Rule # 5 (cont.)

- Let's look at how things work when we don't recombine the output of the SG_Y device:



- The state vector emerging from the SG_Y device is

$$|\psi\rangle = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ i\sqrt{\frac{1}{2}} \end{bmatrix}$$

- Now the probability of emerging from the + and – terminals of the second SG_Z device is:

$$\text{Prob +z} = \left|\langle\psi|+z\rangle\right|^2 = \left|\begin{bmatrix} \sqrt{\frac{1}{2}} \\ i\sqrt{\frac{1}{2}} \end{bmatrix}^* \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|^2 = \left|\sqrt{\tfrac{1}{2}} + 0\right|^2 = \tfrac{1}{2}$$

$$\text{Prob -z} = \left|\langle\psi|-z\rangle\right|^2 = \left|\begin{bmatrix} \sqrt{\frac{1}{2}} \\ i\sqrt{\frac{1}{2}} \end{bmatrix}^* \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right|^2 = \left|0 - i\sqrt{\tfrac{1}{2}}\right|^2 = \tfrac{1}{2}$$

- So when we don't mix…the prob. Is 50-50…and we see how to **derive** that answer using the rules of QM.

Rule #6
Time Evolution Rule

- "It turns out that *any* quanton's state Iψ> can be written as a superposition of energy eigenvectors $IE_1>$, $IE_2>$,… If at t=0 the quanton's state is

$$\left|\psi(0)\right\rangle = c_1\left|E_1\right\rangle + c_2\left|E_2\right\rangle + \ldots$$

the (assuming we leave the quanton alone and do not try to determine any of its observables) at a later time *t*, the quanton's state will be

$$\left|\psi(t)\right\rangle = c_1 e^{-iE_1 t/\hbar}\left|E_1\right\rangle + c_2 e^{-iE_2 t/\hbar}\left|E_2\right\rangle + \ldots$$

where $\hbar = h/2\pi$ (h is Plank's constant) and E_1, E_2… are the energy values associated with the eigenvectors $IE_1>$, $IE_2>$,…, respectively.

- First, this is basically Newton's Second law for QM because it tells us how to determine the future state of the quanton given we know it's initial state.
- Also, remember a couple of important facts about eigenvectors
  - They are normalized
  - They are orthogonal
  - They "span" their space (e.g. x,y,z "span" 3-d space)
  - This implies that we can use them like "unit-vectors" to define the components of any arbitrary state vector.

Rule #6
Example

- Here is an example for rule number 6. First, for this example we will assume that the energy of a quanton is determined by the projection of its spin on the z-axis. If the projection is positive with have $+E_o$ if the projection is negative, we have $-E_o$. So the energy eigenvectors are just the spin-z eigenvectors.

$$|E_o\rangle = |+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |-E_o\rangle = |-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Now it the initial state a t=0 is just the +x eigenvector we have

$$|\psi(0)\rangle = |+x\rangle = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix} = \sqrt{1/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sqrt{1/2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- At a later time t we get

$$|\psi(t)\rangle = \sqrt{1/2}e^{-iE_o t/\hbar}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sqrt{1/2}e^{+iE_o t/\hbar}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1/2}e^{-iE_o t/\hbar} \\ \sqrt{1/2}e^{+iE_o t/\hbar} \end{bmatrix}$$

The Wavefunction

- When listing the rules of QM we considered examples that focused on the spin observables. Now let's focus on the spatial observables.
  - More general since position and momentum are not discrete quantities.
  - The "wavefunction" is the state function for the spatial observables.
- The approach taken by Moore in Section 6.4 I find somewhat awkward. I'm going to take a slightly different approach.
- Recall a couple of properties for the spin state:
  - The vectors are normalized
    $$\left|\langle u|u\rangle\right|^2 = 1$$
  - We want to use these to make a statement about the probability that the quanton is in a particular state.
  - It has a finite number of complex components.
  - Probability Rule: $P(a) = \left|\langle \psi_o | A\rangle\right|^2$
- As the name implies we want to use a function
  - It need to be continuous (well behaved)
  - It needs to be normalize (more in a minute)
  - It is typically complex.

$$\psi(x)$$

Wave Function

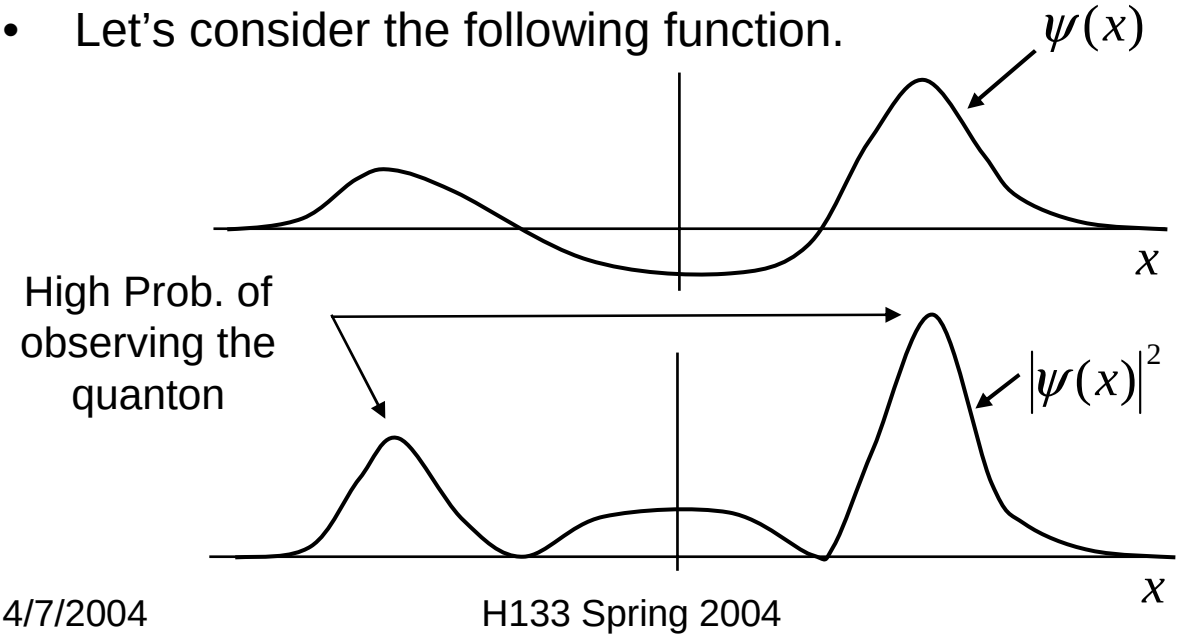
- The normalization condition is expressed as

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx = 1$$

- Notice that this implies we cannot use any function, the integral has to be finite (when can always “normalize” it to one.)
- What does this function represent?
  - The probability of finding a particle between x_1 and x_2:

$$P(x_1 \le x \le x_2) = \int_{x_1}^{x_2} |\psi(x)|^2 dx$$

  - So the square of the wave function is a *probability density.*
  - The wave function itself is then something like the square root of the *probability density.*
- Let’s consider the following function.



Free Particle

- What does the wave function of a free particle look like? Let's assume that the particle has a definite momentum.
  - This implies that the state should be an eigenfunction of momentum. In 1-dimension this is

$$\psi_{p_o}(x) = Ae^{ip_o x/\hbar} = A(\cos(p_o x/\hbar) + i\sin(p_o x/\hbar))$$

  - Notice that this is just an oscillating complex wave with amplitude A. The function will repeat when $p_o x/h$ increases by 2π. This represents one wavelength, λ.

$$2\pi = p_o \lambda/\hbar \Rightarrow \lambda = \frac{2\pi\hbar}{p_o} = \frac{h}{p_o}$$

  - Notice that this is just the de Broglie wavelength! Things are hanging together. We can also use some definitions from the past:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{p}{\hbar} \qquad E = hf = \hbar\omega \Rightarrow \omega = \frac{E}{\hbar}$$

- Since the energy of a free particle is just $E = p^2/2m$ and we are in a definite momentum state (eigenfunction) we must also be in an eigenfunction of energy. So now we can use Time-Evolution Rule to determine what the wave function will look like at a later time, t:

$$\psi_{p_o}(x,t) = Ae^{ip_o x/\hbar}e^{-iEt/\hbar} = Ae^{i(p_o x - Et)/\hbar}$$

$$= Ae^{i(kx-\omega t)} = A\left(\cos(kx-\omega t) + i\sin(kx-\omega t)\right)$$

Traveling Wave!

2-Slit Interference

- Now let's return to the 2-slit interference experiment and understand what we experimentally observed. First a couple of important statements
  - The particles (photons, electrons, etc.) are "Free Particles" and their wave functions are an eigenfunction of momentum (monchromatic, monoenergetic).
  - 2-Slit interference requires a 2 or 3 dimensional wave so we need to generalize our wave function from the previous discussion

$$\psi_{p_o}(r,t) = A(r)e^{i(kr-\omega t)}$$

  - The amplitude A(r) depends on r because as the wave spreads out it will loose amplitude (energy conservation or probability conservation).
- So we don't have to solve both diffraction and interference at the same time, let's assume the slit width is so narrow, that the wave functions are diffracted equally in all directions. (Each slit acts as a true point source).
- If we don't know which slit the photon passed through, the wave function beyond the slit is the superposition:

$$\psi(r_1,r_2,t) = A(r_1)e^{i(kr_1-\omega t)} + A(r_2)e^{i(kr_2-\omega t)}$$

Assume slits are a symmetric

2-Slit Interference

- Now let's assume the screen is a great distance from the two slits so that $A(r_1) \sim A(r_2)$:

$$\psi(r_1, r_2, t) \approx A(r)\left\{e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)}\right\}$$

- Now since our detector makes a position measurement the wave function will collapse and give us a hit in a particular location. The probability that the hit is located in some region is given by integrating the probability density:

$$\begin{aligned}
\left|\psi(r_1, r_2, t)\right|^2 &= \left|A(r)\right|^2 \left\{e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)}\right\}^* \left\{e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)}\right\} \\
&= \left|A(r)\right|^2 \left\{e^{-i(kr_1-\omega t)} + e^{-i(kr_2-\omega t)}\right\}\left\{e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)}\right\} \\
&= \left|A(r)\right|^2 \left\{1+1+e^{-i(kr_2-\omega t)}e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)}e^{-i(kr_1-\omega t)}\right\} \\
&= \left|A(r)\right|^2 \left\{2+\left(e^{ik(r_1-r_2)} + e^{-ik(r_1-r_2)}\right)\right\} \\
&= \left|A(r)\right|^2 \left\{2+2\cos\left[k(r_1-r_2)\right]\right\}
\end{aligned}$$

- The $r_1 - r_2$ term is just the path difference to the point on the screen…recall that this is equal to $d\sin\theta$ and k can be expressed in terms of wavelength.

$$\left|\psi(r_1, r_2, t)\right|^2 = \left|A(r)\right|^2 \left\{2+2\cos\left[2\pi\left(\frac{d\sin\theta}{\lambda}\right)\right]\right\}$$

Peaks when

$d\sin\theta = n\lambda$

Const. Interf. Cond.

2-Slit Interference (Destroyed)

- Can we discuss what happens when we place detectors near the slits. (We saw "experimentally" that the interference pattern disappeared.)
- When we put detectors near the slit, the wave function is forced to collapse at that stage of the experiment. If the detector by slit #1 registers then the wave function after slits becomes:

$$\psi(r_1,t) = A(r_1)e^{i(kr_1-\omega t)}$$

- And the probability density for the photon to hit any place becomes:

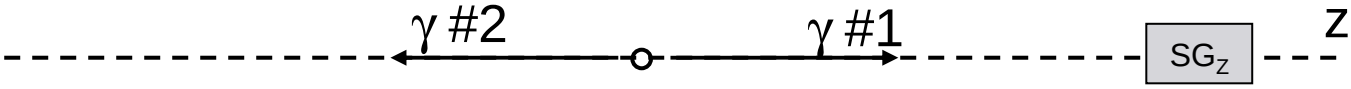
$$\left|\psi(r_1,t)\right|^2 = \left|A(r_1)\right|^2\left|e^{i(kr_1-\omega t)}\right|^2 = \left|A(r_1)\right|^2\cdot 1 = \left|A(r_1)\right|^2$$

- This is just a smooth function…displaying NO interference pattern. This is consistent with our results from chapter 5.

In QM particles have wave functions which are related to the probability of an observable. The different parts of the wave functions can interfer with each other

Collapse of Wave function

- The collapse of the wave function is perhaps one of the strangest features of QM. Read section 6.5 in Moore.
- One important point that I want to emphasis from that section is the idea that the detector "forces" the wave function to collapse. However, as Moore points out this collapse is not tied to the details of the detector. In fact there is a simple (and strange example) of wave function collapse to illustrate this point.
- Example:
  - Pion (type of particle) decays to two photons.



  - Each photon can have a spin aligned (+ or -) with respect to the z-axis.
  - Because of angular momentum conservation, if photon 1 has +, then photon #2 has "–" and vice versa.
  - Before we make any measurements, both photons have some probability of being either + or -.
  - Now Photon #1 enters our detector and we measure it to have spin +…its wave function has collapsed.
  - The wave function for photon #2 also collapses! Even though it may be on the other side of the galaxy, far from our detector!