(Pre-)Algebras for Linguistics

6. Modelling Worlds

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Linguistics 680:
Formal Foundations

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Some Special Kinds of Subsets of Preorders

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  - a **proper filter** iff it is a filter and $S \neq P$
  - a **maximal filter** iff it is a proper filter but not a proper subset of a proper filter.

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- Let's see why this is so.

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- Part of the proof is this Corollary of Stone's Lemma: $p \sqsubseteq q$ iff every ultrafilter containing p also contains q.

Modeling Possible Worlds

These theorems justify modelling possible worlds as the ultrafilters of the preboolean algebra of propositions preordered by entailment (discuss).