(Pre-)Algebras for Linguistics

2. Introducing Preordered Algebras

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Linguistics 680:
Formal Foundations

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  - **Semilattices**: $\circ$ is associative, commutative, and idempotent.
- A **monoid** is a semigroup with a two-sided identity element $\mathbf{e}$.

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  - The two definitions coincide in the unary case.
  - a binary operation $\circ$ is tonic iff (1) for each a, the function that maps each b to $a \circ b$ is tonic, and (2) for each b, the function that maps each a to $a \circ b$ is tonic.

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- An operation in a preordered algebra is said to have a property **up to equivalence (u.t.e.)** if it holds with $=$ replaced by $\equiv$, where $\equiv$ is the equivalence relation induced by the preorder.
- For example, $\circ$ is commutative u.t.e. iff for all a and b, $a \circ b \equiv b \circ a$.

Some Kinds of Preordered Algebras

For future reference:

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- Note: A 'prewidget' is a widget if it is antisymmetric, but not otherwise!

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- For example, in the binary case, this means that if $a \equiv b$ and $c \equiv d$, then $a \circ c \equiv b \circ d$.

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An Important Example of an Ordered Monoid

For any set A, we have seen that $\wp(A^*)$ forms a monoid with

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- $\bullet$ (language concatenation) as the binary operation
- 1_A as the two-sided identity.

We turn this into an ordered monoid by taking the order to be subset inclusion of languages. (You need to check that $\bullet$ is monotonic in both arguments.)

Two Important Examples of an Ordered Semilattice

In both examples, we take the order to be the subset inclusion ordering on $\wp(A)$, for some set A.

- Example 1: take the binary operation to be set intersection. Observation: $a \subseteq b$ iff $a \cap b = a$.

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- Example 1: take the binary operation to be set intersection. Observation: $a \subseteq b$ iff $a \cap b = a$.
- Example 2: take the binary operation to be set union. Observation: $a \subseteq b$ iff $a \cup b = b$.

These observations motivate the following definitions.

Two Kinds of Presemilattices

Suppose $\langle P, \sqsubseteq, \circ \rangle$ is a presemilattice, i.e. $\circ$ is monotonic in both arguments, associative u.t.e., commutative u.t.e., and idempotent u.t.e. Then it is called:

- **upper** iff, for all $a, b \in P$, $a \sqsubseteq b$ iff $a \circ b \equiv b$.

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- **upper** iff, for all $a, b \in P$, $a \sqsubseteq b$ iff $a \circ b \equiv b$.
- **lower** iff, for all $a, b \in P$, $a \sqsubseteq b$ iff $a \circ b \equiv a$.

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- In a lower presemilattice, $\circ$ is a glb operation. (hence usually written $\sqcap$).

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Suppose $\langle P, \sqsubseteq, \circ \rangle$ is a preorder with a (not necessarily tonic) binary operation. Then

- if $\circ$ is a lub operation, then in fact $\langle P, \sqsubseteq, \circ \rangle$ is an upper presemilattice.

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- if $\circ$ is a lub operation, then in fact $\langle P, \sqsubseteq, \circ \rangle$ is an upper presemilattice.
- if $\circ$ is a glb operation, then in fact $\langle P, \sqsubseteq, \circ \rangle$ is a lower presemilattice.

The Semantics of *and* and *or*

We can use presemilattices to analyze the meanings of the two English words *and* and *or* (in-class exercise).