(Pre-)Algebras for Linguistics

5. Prelattices

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Linguistics 680:
Formal Foundations

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- Absorption u.t.e.:

$$(p \sqcup q) \sqcap q \equiv q \equiv (p \sqcap q) \sqcup q;$$

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- **Theorem**: a prelattice is distributive u.t.e. iff the following equivalence holds for all $a, b, c \in P$ (obtained from the one above by interchanging $\sqcap$ and $\sqcup$):

$$p \sqcup (q \sqcap r) \equiv (p \sqcup q) \sqcap (\mathrm{p} \sqcup r)$$

(Pseudo-)Complement Operations

- Suppose $\langle P, \sqsubseteq, \sqcap, \rightarrow, \bot \rangle$ is a heyting presemilattice with a bottom element $\bot$. Then a unary operation $\neg$ on P is called a **pseudocomplement** operation iff, for all $p \in P$,

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- A pseudocomplement operation is called a **complement** operation provided, for all $a \in P$, $\neg(\neg a) \sqsubseteq a$, so that in fact $\neg(\neg a) \equiv a$.

Pre-Heyting and Pre-Boolean Algebras

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- Special case of preceding: $2 = \wp(1) = \{0, 1\}$. Semanticists often call this the algebra of **truth values**, and rename 1 and 0 to $\mathbf{t}$ and $\mathbf{f}$ respectively.

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- Sentences with the same truth conditions have the same meaning.

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- So which subsets of P should be in W?
- To answer this, we need to know more about certain special subsets of preboolean algebras.