(Pre-)Algebras for Linguistics

1. Review of Preorders

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Linguistics 680:
Formal Foundations

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(Pre-)Orders and Induced Equivalence

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- The equivalence relation $\equiv$ **induced** by the preorder is defined by $a \equiv b$ iff $a \sqsubseteq b$ and $b \sqsubseteq a$.
- If $\sqsubseteq$ is an order, then $\equiv$ is just the identity relation on A, and correspondingly $\sqsubseteq$ is read as 'less than or equal to'.

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- $a \in A$ (not necessarily $\in S$)

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Note: the definition of greatest/least above is equivalent to the one in Chapter 3.

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- All greatest (least) members of S are equivalent.
- And so all tops (bottoms) of A are equivalent.
- And so if $\sqsubseteq$ is an order, S has at most one greatest (least) member, and A has at most one top (bottom).

LUBs and GLBs

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- A lub (glb) of A is the same thing as a top (bottom).
- A lub (glb) of $\emptyset$ is the same thing as a bottom (top).

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- If A has a unique top (bottom), it is written $\top$ ($\bot$).

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Monotonicity, Antitonicity, and Tonicity

Suppose A and B are preordered by $\sqsubseteq$ and $\leq$ respectively. Then a function $f\colon A \to B$ is called:

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- **tonic** iff it is either monotonic or antitonic.

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- Two preordered sets are said to be **preorder-isomorphic** provided there is a preorder isomorphism from one to the other.