# Assumptions of (a) Set Theory

Carl Pollard
Ohio State University

Linguistics 680
Formal Foundations
Thursday, September 23, 2010

**These slides are available at:**

`http://www.ling.osu.edu/∼scott/680`

(1) **Sets and Membership**

- We assume there exist things called **sets**.
- We assume there is a relationship, called **membership**, which, for any sets $A$ and $B$, either does or does not hold between them.
- If it does, we say $A$ is a member of $B$, written $A \in B$.
- If it doesn't, we say $A$ is not a member of $B$, written $A \notin B$.
- We never say what sets are: they are the *unanalyzed primitives* of set theory and cannot be reduced to, or explained in terms of, more basic things that are not sets.

## (2) **Assumptions about the Membership Relation**

- We make certain further assumptions *about* membership.
- Our set theory consists of these additional assumptions plus any conclusions we can draw from them using valid reasoning.
- For now, we'll state these assumptions informally in English.
- Later we'll state them more precisely in a special symbolic language (the language of set theory), and the precise restatements of the assumptions will be called the *axioms* of our set theory.
- Also for now we don't say exactly what counts as valid reasoning.
- Later, we'll specify what counts as valid reasoning in terms of mathematical objects called *formal proofs* which deduce new sentences (in the language of set theory) from the axioms.
- There is nothing privileged about *our* set theory; there are other set theories which start with different assumptions.

(3) **Assumption 1 (Extensionality)**
If $A$ and $B$ have the same members, then they are the same set (written $A = B$).

- We don't mention that $A$ and $B$ are sets, because we're doing set theory (so the only things we are talking about are sets).
- We needn't *assume* that if $A$ and $B$ do *not* have the same members, then they are *not* the same set (written $A \neq B$). That's because, if they were the same set, then everything about them, including what members they had, would be the same.
- If every member of $A$ is a member of $B$, we say that $A$ is a **subset** of $B$, written $A \subseteq B$.
- If $A \subseteq B$, $B$ might have members that are not in $A$. In that case we say $A$ is a **proper** subset of $B$, written $A \subsetneq B$.
- But if both $A \subseteq B$ and $B \subseteq A$, then it follows from Extensionality that $A = B$.

(4) **Assumption 2 (Empty Set)**
There is a set with no members.

- From this assumption together with Extensionality we can conclude that the there is *only* one set with no members.
- We call this set the **empty** set, written '$\emptyset$'.
- But later, we'll sometimes write it as '0'.
- That's because in the usual way of doing arithmetic within set theory (covered in Chapter 4) zero *is* the empty set.
- As yet, we have no basis for concluding that there are any sets other than the empty set.
- For example, we are not even able to make a valid argument that there is a set with $\emptyset$ as its only member.
- We remedy this situation by making a few more assumptions.

(5) **Assumption 3 (Pairing)**
For any sets $A$ and $B$, there is a set whose members are $A$ and $B$.

- Even though we say 'sets' here, we don't mean to rule out the possibility that $A$ and $B$ are the same set.
- Because of Extensionality again, there is *only* one set whose members are $A$ and $B$, which we write as $\{A, B\}$, or $\{B, A\}$.
- More generally, we will notate any nonempty finite set by listing its members, separated by commas, between curly brackets, in any order.
- We'll get clear about what we mean by 'finite' in Chapter 5, but for now we'll just rely on intuition.
- In listing the members of a set, repititions don't count, so e.g. if $A$ and $B$ are the same set, then $\{A, B\}$ is the same set as $\{A\}$.
- So it makes no sense to talk about *how many times* $A$ is a member of $B$: either it is or it isn't.

(6) **Definition (Singleton)**
A **singleton** is a set with only one member.

- For any set $A$,we have enough resources now to prove informally (i.e. make a valid argument in English) that there is a singleton setwhose member is $A$. (Of course this is written '$\{A\}$'.
- One singleton set is the set $\{0\}$ whose member is 0.
- $\{0\}$ is also written '1', because in the usual way of doing arithmetic within set theory, it is the same as the number one.

(7) **Preview of the Natural Numbers**

- As mentioned above, we'll define 0 to be $\emptyset$, and 1 to be $\{0\}$.
- By Pairing, we know there is a set, $\{0, 1\}$, whose only members are 0 and 1. Let's say that this is what the number 2 is.
- There seems to be a pattern here, in which the next step would be to say that 3 is the set whose only members are 0, 1, and 2.
- But as yet we don't have sufficient resources to prove that there *is* such a set!
- To say nothing of proving that there is a set whose members are the natural numbers.
- In fact, as yet we don't even know what 'natural number' means.
- But soon we will (Chapter 4).

(8) **Assumption 4 (Union)**

For any set $A$, there is a set whose members are those sets which are members of (at least) one of the members of $A$.

- Extensionality ensures the uniqueness of such a set, which is called the **union** of $A$, written $\bigcup A$.
- If $A = \{B, C\}$, then $\bigcup A$ is the set each of whose members is in either $B$ or $C$ (or both), usually written $B \cup C$.
- In general, $B \cup C$ is *not* the same thing as $\{B, C\}$!

(9) **Definition (Successor)**

For any set $A$, the **successor** of $A$, written $s(A)$, is the set $A \cup \{A\}$.

- That is, $s(A)$ is the set with the same members as $A$, except that $A$ itself is also a member of $s(A)$.
- Nothing in our set theory will rule out the possibility that $A \in A$, in which case $s(A) = s(A)$.
- However, some widely used set theories include an assumption (called **Foundation**) which does rule out this possibility.
- For example, we can prove that 1 is the successor of 0, and that 2 is the successor of 1.
- Now we can say what 3 is: the successor of 2!
- Likewise we can say what 4, 5, etc. are.
- But we still can't say exactly what we mean by a natural number.

(10) **Assumption 5 (Powerset)**
For any set $A$, there is a set whose members are the subsets of $A$.

- Again, Extensionality guarantees the uniqueness of such a set, which we call the **powerset** of $A$, written $\wp(A)$.
- In general, $\wp(A)$ is not the same set as $A$, because usually the subsets of a set are not the same as the members of the set.
- For example, 0 is a subset of 0 (in fact, every set is a subset of itself), but obviously 0 is not a member of 0 (since 0 is the empty set).

(11) **Assumptions vs. Definitions**

a. Notice a crucial difference between the successor of a set $A$ and the powerset of $A$: successor is *defined* in terms of things whose existence can already be established on the basis of previous assumptions (singletons, unions), whereas the existence of the powerset of $A$ is *assumed*.

b. Why didn't we just *define* $\wp(A)$ to be the set whose members are the subsets of $A$?

c. It's because nobody has found a valid argument (based on just the first four assumptions) that there *is* such a set!

d. More generally, there is no guarantee that, for an arbitrary condition on sets $P[x]$, there is a set whose members are all the sets $x$ such that $P[x]$.

e. But this fact did not become known till 1902.

(12) **Russell's Paradox**

a. Let $P[x]$ be the condition '$x$ is not a member of itself'.

b. Following Russell, we will show that there cannot be a set whose members are all the sets $x$ such that $P[x]$.

c. Suppose $R$ were such a set.

d. Then either (i) $R$ is a member of itself, or (ii) it isn't.

   i. Suppose $R$ *is* a member of itself, Then it cannot be a member of $R$, since the members of $R$ are sets which are *not* members of themselves. But then it is *not* a member of itself.

   ii. Suppose $R$ is *not* a member of itself. Then it must be in $R$. But then, it *is* a member of itself.

   iii. Either way, assuming (c) leads to a contradiction.

e. So the assumption (c) must have been false.

## (13) **An Imaginable Set-Theoretic Assumption Bites the Dust**

- Russell's Paradox shows we don't have the option of adding the following to our set theory:

  **Tentative Assumption (Comprehension)**

  For any condition $P[x]$ there is a set whose members are all the sets $x$ such that $P[x]$.

- The following assumption is usually adopted instead.

(14) **Assumption 6 (Separation)**
For any set $A$ and any condition $P[x]$, there is a set whose members are all the $x$ in $A$ that satisfy $P[x]$.

- So far, assuming Separation has not been shown to lead to a contradiction.
- Separation is so-called because, intuitively, we are separating out from $A$ some members that are special in some way, and collecting them together into a set.
- By Extensionality, there can be only one set whose members are all the sets $x$ in $A$ that satisfy $P[x]$.
- We call that set $\{x \in A \mid P[x]\}$.

## (15) **Intersection**

- In naive introductions to set theory, the **intersection** of two sets $A$ and $B$, written $A \cap B$, is often 'defined' as the set whose members are those sets which are members of both $A$ and $B$.
- But how do we know there is such a set?
- If we assume Separation and take $P[x]$ to be the condition $x \in B$, then we can (unproblematically) define $A \cap B$ to be $\{x \in A \mid x \in B\}$.
- $A$ and $B$ are said to **intersect** provided $A \cap B$ is nonempty.
- A set $A$ is called **pairwise disjoint** if no two distinct members of it intersect.

(16) **Set Difference**

- For two sets $A$ and $B$, if we take $P[x]$ to be the condition $x \notin B$, then Separation guarantees the existence of the set $\{x \in A \mid x \notin B\}$.
- This set is called the **set difference** of $A$ and $B$, or alternatively the **complement** of $B$ **relative to** $A$, written $A \setminus B$.

## (17) **Nonexistence of a Universal Set**

- A set is called **universal** if every set is a member of it.
- We can prove in our set theory that there is no universal set.
- For suppose $A$ were a universal set. Let $P[x]$ be the condition $x \notin x$. Then by Separation, there must be a set $\{x \in A \mid x \notin x\}$. But Russell's argument showed that there can be no such set. So the assumption that there was a universal set must have been false.

(18) **Ordered Pairs**

- If $A$ and $B$ are sets, we call the set $\{\{A\}, \{A, B\}\}$ the **ordered pair** of $A$ and $B$, also written $\langle A, B\rangle$.
- $\langle A, B\rangle$ differs from $\{A, B\}$ in the crucial respect that no matter what $A$ and $B$ are, $\{A, B\} = \{B, A\}$, but $\langle A, B\rangle = \langle B, A\rangle$ only if $A = B$.
- More generally, if $A$, $B$, $C$, and $D$ are sets, then $\langle A, B\rangle = \langle C, D\rangle$ only if $A = C$ and $B = D$.
- If $C$ is the ordered pair of $A$ and $B$, $A$ is called the **first component** of $C$, and $B$ is called the **second component** of $C$.

## (19) **The Cartesian Product of Two Sets**

- For any sets $A$ and $B$, there is a set whose members are all those sets which are ordered pairs whose first component is in $A$ and whose second component is in $B$. (It's instructive to try to prove this. Hint: use Separation.)
- By Extensionality there can be only one such set. It is called the **cartesian product** of $A$ and $B$, written $A \times B$.
- For any sets $A$, $B$, $C$, and $D$, $A \times B = C \times D$ only if $A = C$ and $B = D$. (Try to prove this.)
- $A$ is called the **first factor** of $A \times B$, and $B$ the **second factor**.

(20) **Ordered Triples**

- The **ordered triple** of $A$, $B$, and $C$, written $\langle A, B, C \rangle$, is defined to be the ordered pair whose first component is $\langle A, B \rangle$ and whose second component is $C$.
- Then $A$, $B$, and $C$ are called, respectively, the **first**, **second**, and **third components** of $\langle A, B, C \rangle$.
- The **(threefold) cartesian product** of $A$, $B$, and $C$, written $A \times B \times C$, is defined to be $(A \times B) \times C$. This is the set of all ordered triples whose first, second, and third components are in $A$, $B$, and $C$ respectively.
- The definitions can be extended to ordered quadruples, quintuples, etc., and to $n$-fold cartesian products for $n > 3$, in an obvious way.

(21) **Definitions (Cartesian Powers)**

For any set $A$, a **cartesian power** of $A$ is a cartesian product all of whose factors are $A$.

- The **first cartesian power** of $A$ is just $A$, also written $A^{(1)}$.
- The **cartesian square** of $A$, written $A^{(2)}$, is $A \times A$.
- The **cartesian cube** of $A$, written $A^{(3)}$, is $A \times A \times A$
- More generally, for $n > 3$, the $n$**-th cartesian power** of $A$, written $A^{(n)}$, is the $n$-fold cartesian product all of whose factors are $A$.
- Additionally, the **zero-th cartesian power** of $A$, $A^{(0)}$, is defined to be the set 1 $(= \{\emptyset\})$.
- This last definition is closely related to the arithmetic fact that for any natural number $n$, $n^0 = 1$, but we postpone the explanation.