1. Estimate the order of magnitude of the degeneracy temperature in Kelvin or eV or any other unit which seems most natural for the following systems:
   (a) Dilute alkali gas of $^{87}\text{Rb}$ atoms in BEC experiments;
   (b) liquid Helium 4;
   (c) electron gas in metals;
   (d) neutrons in a neutron star.

2. I use the following Fourier transform (F.T.) conventions for a system with volume $\Omega$. For the creation and annihilation operators

   \[ a_k = \frac{1}{\sqrt{\Omega}} \int d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \psi(\mathbf{r}); \quad \psi(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} \sum_k e^{+i\mathbf{k} \cdot \mathbf{r}} a_k \]

   These can be either bosons or fermions (with the spin index omitted for simplicity). For all other operators

   \[ B_q = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} B(\mathbf{r}); \quad B(\mathbf{r}) = \frac{1}{\Omega} \sum_q e^{+i\mathbf{q} \cdot \mathbf{r}} B_q. \]

   In addition: $\int d\mathbf{r} e^{+i\mathbf{q} \cdot \mathbf{r}} = \Omega \delta_{\mathbf{q},0}$.

   (a) The “first quantized” form of the current operator is given by

   \[ J(\mathbf{r}) = \frac{1}{2m} \sum_{i=1}^{N} \left[ \mathbf{p}_i \delta(\mathbf{r} - \mathbf{r}_i) + \delta(\mathbf{r} - \mathbf{r}_i) \mathbf{p}_i \right] \]

   Using the basis set $|\mathbf{r}_1 \ldots \mathbf{r}_N\rangle = (N!)^{-1/2} \psi^\dagger(\mathbf{r}_1) \ldots \psi^\dagger(\mathbf{r}_N)|0\rangle$, or otherwise, show that the “second quantized” form is given by

   \[ J(\mathbf{r}) = \frac{-i}{2m} \left[ \psi^\dagger(\mathbf{r}) \nabla \psi(\mathbf{r}) - \left( \nabla \psi^\dagger(\mathbf{r}) \right) \psi(\mathbf{r}) \right]. \]
(b) Show that the F.T. of the above result is given by

\[ J_q = \sum_k \frac{k}{m} a_{k-q}^\dagger a_{k+q}. \]

(c) Show that the kinetic energy operator \( T = \sum_k \frac{k^2}{2m} a_k^\dagger a_k \) can be written as

\[ T = -\frac{1}{2m} \int dr \psi^\dagger(r) \nabla^2 \psi(r) \]

(d) Show that the potential energy operator \( V = \frac{1}{2} \int dr \int dr' V(r-r') \psi^\dagger(r) \psi^\dagger(r') \psi(r') \psi(r) \)

can be written as

\[ V = \frac{1}{2\Omega} \sum_{k,k',q} V_q a_{k+q/2}^\dagger a_{k'-q/2}^\dagger a_{k'+q/2} a_{k-q/2} \]

3. This problem will take you through the details of the derivation of the Kubo formula for \( T \neq 0 \) (which were skipped in class). The idea is very similar to that used in class for the \( T = 0 \) derivation, except now one needs to find the full density matrix to first order in the perturbation (in place of just the ground state wave-function).

Start with the equation of motion for the density matrix \( \rho \) given by

\[ i \frac{d}{dt} \rho = [\mathcal{H}_{\text{total}}, \rho] \]

where \( \mathcal{H}_{\text{total}} = \mathcal{H} + \mathcal{H}' \) where \( \mathcal{H} \) is the Hamiltonian of the interacting many-body system and \( \mathcal{H}' = -AF_A(t) \) is the coupling of the external perturbation \( F_A(t) \) to the system through the operator \( A \). The initial condition is given by the fact that the system started out in thermal equilibrium at a temperature \( T = \beta^{-1} \) in the distant past:

\[ \rho(t \to -\infty) = \exp(-\beta \mathcal{H})/Z \equiv \rho_0, \quad Z = \text{Tr} \{ \exp(-\beta \mathcal{H}) \}. \]

(a) Switch to the interaction representation in which all operators are transformed according to \( \tilde{Q}(t) = \exp(i\mathcal{H}t)Q \exp(-i\mathcal{H}t) \). Show that the interaction representation equation of motion is given by

\[ i \frac{d}{dt} \tilde{\rho}(t) = [\tilde{\mathcal{H}}'(t), \tilde{\rho}(t)] \]
(b) Solve the equation of motion of part (a) subject to the given initial condition and show that, to first order in the perturbation, the result is

$$\tilde{\rho}(t) = \rho_0 + i \int_{-\infty}^{t} dt' \left[ \tilde{\rho}_0, \tilde{H}'(t') \right] + \ldots$$

(c) We finally want to calculate the response $\langle B \rangle(t) \equiv \text{Tr} \{ \rho(t)B \}$.

First, show that this may be written in the interaction representation as $\langle B \rangle(t) = \text{Tr} \{ \tilde{\rho}(t)\tilde{B}(t) \}$. Hint: use the cyclic property of the trace $\text{Tr} \{ABC\} = \text{Tr} \{BCA\}$.

(d) Now assume (for simplicity) that $\text{Tr} \{ \rho_0B \} = 0$ and combine the results obtained above to conclude that

$$\langle B \rangle(t) = \int_{-\infty}^{+\infty} dt' \chi(t-t') F_a(t')$$

with

$$\chi(t-t') = i\Theta(t-t') \left\langle [\tilde{B}(t), \tilde{A}(t')] \right\rangle_T$$

where $\langle \ldots \rangle_T$ represents an average with respect to the equilibrium density matrix $\rho_0$ at temperature $T$.

4. Consider the classical damped, harmonic oscillator whose equation of motion is given by

$$\ddot{x} + \gamma \dot{x} + \Omega^2 x = f(t)/m$$

Here $x(t)$ is the displacement, $m$ the mass of the particle, $\Omega$ the natural frequency in the absence of damping, $\gamma$ the damping coefficient and $f(t)$ the external force. Use the F.T. convention

$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} x(\omega)$$

(a) Find the complex susceptibility $\chi(\omega)$ which describes the linear response $x$ of the system to the external force $f$, i.e., $x(\omega) = \chi(\omega)f(\omega)$.

(b) Where in the complex-$z$ plane are the poles of $\chi(z)$ located? What is their physical meaning?

(c) Calculate $\text{Re}\chi(\omega)$ and $\text{Im}\chi(\omega)$ and plot these functions for $\gamma \ll \Omega$ after determining the even/odd properties of these function under $\omega \rightarrow -\omega$.

(d) Show that for a forcing function $f(t) = F_0 \cos(\omega t)$, the steady-state response of the system is of the form $x(t) = F_0 A(\omega) \cos(\omega t - \phi(\omega))$ where
$A(\omega) \geq 0$. Find the amplitude $A(\omega)$ and the phase $\phi(\omega)$. How are they related to $\chi(\omega)$?

(e) The rate at which the external force does work on the system is given by $dW/dt = f(t)\dot{x}(t)$. Consider the periodic external force of part (d). In the steady state the average power dissipated can be determined by time-averaging $dW/dt$ over a cycle with period $2\pi/\omega$. Calculate the average power dissipated in the steady state and show that

$$\omega \text{Im}\chi(\omega) \geq 0.$$ 

(f) Determine the real time response function $\chi(t)$ by calculating the F.T. of $\chi(\omega)$ using contour integration. You must carefully justify the choice of contours in the complex plane by making sure that the integrand vanishes for large $|z|$ so as to ensure convergence. Comment how this calculation shows that causality (in time) is closely tied to analyticity in the upper half plane (of the complex-$\omega$ plane).