PHYSICS 827

Home Work Assignment # 4 10/15/2010

Due: Fri., Oct. 22, 2010 (by 5:00 PM in the grader mail box).

The exercise numbers are from your text book: Shankar (2nd edition).

1. Ex. 4.2.1, page 129.

2. Ex. 4.2.2, page 139.

3. Ex. 4.2.3, page 139.

4. Show that if the variance of a Hermitian operator Ω vanishes in a state $|\psi\rangle$, then $|\psi\rangle$ must be an eigenvector of Ω.
In other words, let $\langle\psi|\Omega|\psi\rangle = \omega$, then zero variance implies $\langle\psi|\Omega^2|\psi\rangle = \omega^2$.
Now you need to show that this implies $\Omega|\psi\rangle = \omega|\psi\rangle$.

5. Consider a two-level system described by the Hamiltonian

$$H = \begin{pmatrix} \epsilon & \Delta \\ \Delta & -\epsilon \end{pmatrix}$$

What are the possible outcomes of the measurement of the energy of this system? What can you say about the state of the system immediately after its energy is measured?

6. Define the *projection operator* $\mathcal{P} = |\psi\rangle\langle\psi|$ where $\langle\psi|\psi\rangle = 1$.
More generally, $\mathcal{P} = \sum_{n=1}^{K} |n\rangle\langle n|$ is the projection operator onto a K-dimensional linear subspace of a D-dimensional Hilbert space spanned by an orthonormal set $|n\rangle$ (with $n = 1, \ldots, D$ and $1 \le K \le D$).
(a) Show that $\mathcal{P}$ is Hermitian. (b) Show that $\mathcal{P}^2 = \mathcal{P}$ and comment on what this means. (c) What are the eigenvalues and eigenvectors of $\mathcal{P}$?

7. The Pauli matrices are defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the operator $s_n \equiv \hat{\mathbf{n}}\cdot\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ where the unit vector $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ points in the (θ, ϕ)-direction.

(a) Write the operator s_n as a matrix in the basis of normalized eigenstates $|+\rangle$ and $|-\rangle$ of σ_z and check that it is Hermitian. Here we use an obvious notation $|\pm\rangle$ for the eigenstates of σ_z corresponding to eigenvalues ± 1.

(b) Find the eigenvalues of s_n. How do they depend on θ and ϕ?

(c) Show that, with an appropriate choice of the (overall) phase, the eigenvectors of s_n can be written in the form
$|\hat{n}+\rangle = \cos(\theta/2)\exp(-i\phi/2)|+\rangle + \sin(\theta/2)\exp(+i\phi/2)|-\rangle$ and
$|\hat{n}-\rangle = \sin(\theta/2)\exp(-i\phi/2)|+\rangle - \cos(\theta/2)\exp(+i\phi/2)|-\rangle$.
Do not just verify that this solution is correct. Please solve the problem to determine eigenvectors.

(d) Check that $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$ are orthogonal. Also check that these eigenstates reduce to those of the three Pauli matrices when $\hat{\mathbf{n}}$ is chosen to be along the x, y and z directions respectively.

8. Now consider a sequence of experiments that measure the spin-state of atoms with spin-1/2. You do not need to know any of the details of *how* this Stern-Gerlach apparatus works. All you need to know is that the Hermitian operator being measured by an apparatus oriented along the $\hat{\mathbf{n}}$ direction is

$$S_n = (\hbar/2)s_n$$

where s_n was defined in Problem 6. (Below we use the same notation as the previous problem.)

Consider the following set up:
(I) First, prepare a beam of spin-1/2 atoms which are all in the state $|+\rangle$ by passing a beam through a Stern-Gerlach device oriented in the $\hat{\mathbf{z}}$ direction, and keep only those atoms measured to have eigenvalue $+\hbar/2$.
(II) Then, pass the atoms kept in (I) through a second Stern-Gerlach device designed to measure S_n where $\hat{\mathbf{n}}$ lies in the (x, z)-plane, so that $\theta \neq 0$ and $\phi = 0$. Again we retain only those atoms which have eigenvalue $+\hbar/2$.
(III) Finally, pass the atoms which remain through a third Stern-Gerlach experiment that is oriented in the same $\hat{\mathbf{z}}$ direction as the first.

(a) Of all the atoms that entered second device, what fraction are found by the third device to be in the state $|+\rangle$?
(b) What fraction are found by the third device to be in the state $|-\rangle$?
(c) What fraction never made it to the third device?

(d) Your answers will be functions of the angle θ. Argue that your answers make sense for $\theta = 0$ ($\hat{\mathbf{n}} = \hat{\mathbf{z}}$), $\theta = \pi/2 (\hat{\mathbf{n}} = \hat{\mathbf{x}})$ and $\theta = \pi$ ($\hat{\mathbf{n}} = -\hat{\mathbf{z}}$).