Electric field of a solid sphere

1) Volume charge density = \( \rho \) uniform

Find total charge contained in sphere of radius \( R \).

\[
Q_{\text{total}} = \int_{\text{vol of sphere}} \rho \, d\tau
\]

Since \( \rho \) is a constant, we can pull it out of the integral.

\[
\int_{\text{vol of sphere}} d\tau = \text{vol of sphere of radius } R = \frac{4}{3} \pi R^3
\]

\[
Q_{\text{total}} = \frac{4}{3} \pi R^3 \rho
\]
You can do this explicitly also.

Draw volume element in spherical co-ordinate.

Two of the dimensions of $dV$ come from $(dr) (r d\theta)$

Third dimension comes from the azimuthal direction.

\[ \downarrow \]

move along the dotted circle
Third dimension is
$r \sin \theta \, d\phi$

\[ dV = (dr) (r \, d\theta) (r \sin \theta \, d\phi) \]
\[ = r^2 \, dr \, \sin \theta \, d\theta \, d\phi \]

\[ \int dV = \int_0^R r^2 \, dr \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} d\phi \]
\[ = \frac{R^3}{3} \left[ -\cos \theta \right]_0^{\pi} \int_0^{2\pi} d\phi \]
\[ = \frac{4}{3} \pi R^3 \quad \checkmark \]

\[ \cos \theta \bigg|_{\frac{\theta}{\pi}} = 1 - (-1) = 2 \]
Gauss' Law
\[ \oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \]

Electric field outside sphere at point \( r \)

**Step 1: Gaussian Surface**

Make a Gaussian surface of the appropriate symmetry that passes through the point of observation \( r \)

**Step 2: Identify direction and area of electric field**

On the Gaussian surface, the electric field will be constant everywhere and pointing radially outward.

\[ \mathbf{E}(\mathbf{r}) = E \hat{r} \]

for \( \mathbf{r} \) lying on surface \( S \)

**Step 3: Pick an area element**

On the Gaussian surface, the area element will point along the radial direction \( \hat{r} \)

\[ d\mathbf{a} = da \hat{r} \]
Step 4  Find enclosed charge
Since $S$ is outside the sphere the amount of charge enclosed
$= \text{total charge} = \frac{4}{3} \pi R^3 \rho$

Step 5  Put everything together in Gauss's Law.

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{4}{3} \pi R^3 \frac{\rho}{\varepsilon_0}$$

Spherical Gaussian surface of radius $r$

$$E \oint \mathbf{d}a = E \int 4\pi r^2 = \frac{4}{3} \pi R^3 \frac{\rho}{\varepsilon_0}$$

If $r > R$

You can also check $\oint \mathbf{d}a$ explicitly

$$\oint (r d\theta)(r \sin \theta d\phi) = r^2 \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = 4\pi r^2$$
Electric field inside sphere at point \( r \)

1. Gaussian surface \( S \)
2. \( \vec{E}(r) = \vec{E} \hat{r} \)
3. \( d\mathbf{a} = d\mathbf{a} \hat{r} \)
4. Enclosed charge by Gaussian surface

This surface enclosed only part of the charge
\[
Q_{enc} = \oint_S \rho \, d\tau = \rho \int d\tau = \rho \frac{4}{3} \pi r^3
\]

\( \rho \) comes out because constant

\[
E \oint_S d\mathbf{a} = \frac{\rho}{\varepsilon_0} \frac{4}{3} \pi r^3
\]

\[
E \cdot 4\pi r^2 = \frac{\rho}{\varepsilon_0} \frac{4\pi r^3}{3}
\]

\[
\therefore \quad E(r) = \frac{\rho r}{3\varepsilon_0} \hat{r}, \quad r < R
\]
Sketch of electric field along radial dimensions.

Initially the electric field increases as \( r \) increases (but still within the sphere) because more and more charge is enclosed.

For \( r \geq R \) the sphere behave like a point charge concentrated at the center and \( E \propto \frac{1}{r^2} \).
Total charge on inner cylinder

\[ Q_{\text{inner cylinder}} = \int \rho \, d\sigma = \rho \pi a^2 \, L \]

\[ Q_{\text{outer cylinder}} = \int \sigma \, da = \sigma 2\pi b \, L \]

The total system is neutral

\[ Q_{\text{inner}} = Q_{\text{outer}} \]

\[ \rho \pi a^2 \, L = \sigma 2\pi b \, L \]

\[ \rho a^2 = 2\sigma b \]

and \( \sigma \) and \( \rho \) have opposite sign.
Step 1: Gaussian surface: cylinder passing through point $a$.

Step 2: $E = E \hat{s}$

Step 3: $da = da \hat{s}$

Step 4: Enclosed charge:

$Q_{enc} = \oint p \, dr = \frac{p \pi \ell^2}{V_{ol}}$

Step 5: $E \oint_{S}^{a} da = \frac{p \pi \ell^2}{\epsilon_0}$

$E \frac{2 \pi l \ell}{\epsilon_0}$
\[ E = \frac{\rho \delta}{2\varepsilon_0} \]
\[ E(x) = \frac{\rho \delta}{2\varepsilon_0} \quad \text{for} \quad \delta < a \]

**E field** \( a < \delta < b \)

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**Step 1 - 3**

as before

**Step 4:** \( Q_{enc} = \rho \pi a^2 l \)

Notice we have "\( a \)" here and not \( \delta \) because charge exists only up to \( \delta \leq a \)
Step 5:

\[ E \oint da = \frac{\rho \pi a^2 l}{\varepsilon_0} \]

\[ E \ 2\pi bl = \frac{\rho \pi a^2 l}{\varepsilon_0} \]

\[ E(x) = \frac{\rho a^2}{2\varepsilon_0} \hat{s} \]

\[ \text{E field } b > a \]

Step 1-3
Same as before

Step 4
Change enclosed
\[ = 0 \]

(Surface enclosed
both + charge from inner cylinder \( a \)
- charge from outer)
As we cross \( z = b \) the electric field jumps from \( \frac{\rho a^2}{2\varepsilon_0 b} \) to zero.

\[
\Delta E = \text{jump in electric field} = \frac{\rho a^2}{2\varepsilon_0 b}
\]

As shown earlier, \( \rho a^2 = 2\sigma b \)

\[
\Rightarrow \quad \Delta E = \frac{2\sigma b}{2\varepsilon_0 b} = \frac{\sigma}{\varepsilon_0}
\]

\[
\Delta E = \frac{\sigma}{\varepsilon_0} \quad \text{surface charge density}
\]

jump in electric field across a charged surface
# 2.17

Infinite plate of thickness $2d$

volume density $p$.

$E$ field points along $+y$ for $y > 0$

and points along $-y$ for $y < 0$

This fixes the direction of $E$

$E(z)$ can only depend on $y$

because we are told it is an infinite slab

so it cannot have any dependence on $x$ or $z$. Why? because there is no

way to tell one value of $z$ from another.

They are all equivalent for an $\infty$ slab.

(claim for $x$). If there was an edge

we could differentiate a point that is closer

to the edge from one that is further away
E field inside slab \( y < d \)

Step 1: At a point \( y \) away from center, draw the box-like Gaussian surface of area \( A \) and length \( 2y < 2d \).

Step 2: The magnitude of \( E \) is the same on both sides and points along \( \pm \hat{y} \).

Step 3: Area \( \hat{a} \) area points out as shown.
Step 4: Charge enclosed by gaussian surface

\[ \int \vec{P} \, d\mathbf{r} = \text{volume of box} \]
\[ = PA2y \]

Step 5:

\[ \oint \vec{E} \cdot d\mathbf{a} = EA + EA = \frac{PA2y}{\varepsilon_0} \]

contribution from the shaded surfaces.

The other surfaces do not contribute because the area vector is \( \perp \) to \( \vec{E} \) field on that surface.

\[ 2EA = 2 \frac{PAy}{\varepsilon_0} \]

\[ \vec{E}(y) = \frac{Py}{\varepsilon_0} \hat{n} \]

normal to face.
E field outside the box

Step 1-3 same

Step 4: change enclosed = \( P \cdot A \cdot 2d \)

Step 5:

\[ 2EA = \frac{P \cdot A \cdot 2d}{\varepsilon_0} \]

\[ E(y) = \frac{Pd}{\varepsilon_0} \hat{n} \]

Sketch: If \( E \) points along \( +y \) we treat it as positive
If it points along \( -y \) we treat it as negative

Note: as \( d \rightarrow 0 \) and \( 2Pd \rightarrow 0 \)
we get back the field of an infinitely thin slab \( E = \frac{\sigma}{2\varepsilon_0} \)
What is the flux through shaded surface?

We know that if the charge was enclosed by a surface the flux would be

\[ \Phi = \oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{q}{\varepsilon_0} \]

If we choose that surface using symmetry we could find the flux through shaded area.

+ 4 more cubes at the bottom.
What are the total # of exposed faces:

6 sides × \frac{4 \text{ faces}}{\text{side}} = 24 \text{ faces}

The flux through each face is identical (by symmetry)

⇒ flux through any 1 face = \frac{\text{total flux}}{24}

flux through = \frac{9 \epsilon}{24 \epsilon_0}

Brute force:

Electric field of point charge:

\[ E(x) = \frac{q}{4\pi \epsilon_0} \frac{\hat{r}}{r^2} = \frac{q}{4\pi \epsilon_0} \frac{r}{r^3} \]

\[ r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \]

\[ \mathbf{r} = x \hat{x} + y \hat{y} + z \hat{z} \]
\[ \Phi_{\text{Face}} = \int_{\text{area of 1 face}} E \cdot da = \int E \cdot \hat{y} \, dx \, dz \]

 Flux through 1 face

the area of the shaded face points along \( \hat{y} \).

\[ E \cdot \hat{y} = \frac{y}{r^3} \frac{q}{4\pi \varepsilon_0} \bigg|_{\text{evaluated at } y = a} \]

\[ = \frac{q}{4\pi \varepsilon_0} \frac{a}{(x^2 + z^2 + a^2)^{3/2}} \]

\[ \Phi_{\text{Face}} = \frac{q \, a}{4\pi \varepsilon_0} \int_0^a \int_0^a \frac{dz \, dz}{(a^2 + x^2 + z^2)^{3/2}} \]
#1.38  \( a) \quad \mathbf{v}_1 = \frac{r^2}{r} \mathbf{r} \)

\[
\nabla \cdot \mathbf{v}_1 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \mathbf{v}_1 \cdot \mathbf{r} \right)
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^4 \right) = \frac{1}{r^2} \cdot 4r^3 = 4r
\]

Divergence theorem:

\[
\int \nabla \cdot \mathbf{v}_1 \, d\tau = \int_S \mathbf{v}_1 \cdot d\mathbf{a}
\]

LHS:

\[
\int \nabla \cdot \mathbf{v}_1 \, d\tau = \int (4r) \, r^2 \sin \theta \, d\theta \, d\phi \, dr
\]

\[
= (4\pi) \left( 4 \right) \int_0^R r^3 \, dr
\]

\[
\int \nabla \cdot \mathbf{v}_1 \, d\tau = 4\pi R^4
\]

RHS:

\[
\int_S \mathbf{v}_1 \cdot d\mathbf{a} = \int \left( \frac{r^2}{r} \mathbf{r} \cdot \mathbf{r} \right) \, d\mathbf{a}
\]

on surface
of radius \( R \)

\[
= \frac{u_i}{r^2} \, da
\]
\[ \oint \mathbf{v}_1 \cdot d\mathbf{a} = \iiint d\theta \, d\phi \, \sin \theta \, r^4 \]
\[ = R^4 \cdot 4\pi \]
\[ \oint_S \mathbf{v}_1 \cdot d\mathbf{a} = 4\pi R^4 \]

\[ \Rightarrow \text{LHS} = \text{RHS} \quad \text{Div then holds} \]

(b) \[ \mathbf{v}_2 = \frac{1}{r^2} \hat{r} \]
\[ \nabla \cdot \mathbf{v}_2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{r^2} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0 \]

LHS: \[ \int \nabla \cdot \mathbf{v}_2 \, d\tau = 0 \]

RHS: \[ \oint_S \mathbf{v}_2 \cdot d\mathbf{a} = \iint \frac{1}{r^2} \, r^2 \sin \theta \, d\theta \, d\phi \]
\[ = 4\pi \]

LHS \neq \text{RHS} \quad \text{WHY?} \]
The problem is that the calculation of \( \nabla \cdot \mathbf{V}_2 \) is missing a singular contribution at \( r = 0 \).

\[
\nabla \cdot \mathbf{V}_2 = \nabla \cdot \left( \frac{1}{r^2} \mathbf{r} \right) = 4\pi \delta^3 (\mathbf{r})
\]

\[
= \begin{cases} 
0 & r \neq 0 \\
4\pi & r = 0
\end{cases}
\]

Now:

\[
\int \nabla \cdot \mathbf{V}_2 \, d\tau = \int 4\pi \delta^3 (\mathbf{r}) \, d\tau
\]

\[= 4\pi \]

and this agrees with RHS.