LING4400: Lecture Notes 14
Entailment

Contents

14.1 Possible worlds [Carnap, 1947]

World states of intensions let us represent entailment within our logic, of type $\langle\langle s,t\rangle,\langle\langle s,t\rangle,t\rangle\rangle$:

$$\begin{aligned}[\![\mathsf{Entail}]\!]^M &= [\![\lambda_{r:\langle s,t\rangle}\ \lambda_{s:\langle s,t\rangle}\ |\lambda_{w:s}\ r\ w \wedge s\ w| \times 1 = |r|]\!]^M \\ &= [\![\mathsf{All}_{\text{except using s instead of e}}]\!]^M\end{aligned}$$

This is true if all elements of the first (premise) argument are elements of the second (conclusion).

There is a possible world for each combination of functions over each possible domains.

That's a lot, but it's still smaller than the set of possible expressions.

For example, different orderings of conjuncts have the same truth value, so match the same worlds:

$$[\![\mathsf{Intension}\ (\mathsf{Erupts\ Etna} \wedge \mathsf{Erupts\ Wolf})]\!]^M = [\![\mathsf{Intension}\ (\mathsf{Erupts\ Wolf} \wedge \mathsf{Erupts\ Etna})]\!]^M$$

Good, that agrees with our intuition that these expressions 'mean the same thing.'

But there might be cases that are provably the same that people *don't realize* are the same.

For example, the unproven Riehmann Hypothesis about prime numbers may always be true, so:

$$[\![\mathsf{Intension}\ (\mathsf{Equal}\ 2\ 2)]\!]^M = [\![\mathsf{Intension\ RiehmannHypothesis}]\!]^M$$

So, knowing that $2 = 2$ is the same as knowing the Riehmann Hypothesis (if it's true).

(One way around this is to limit the complexity of proof allowed in entailment. We won't though.)

14.2 Propositional entailment

Recall the truth table for And:

$$[\![\mathsf{And}]\!]^M = \begin{array}{|ll|} \hline \text{input} & \text{output} \\ \hline \textbf{False}: & \begin{array}{|ll|} \hline \text{input} & \text{output} \\ \hline \textbf{False}: & \textbf{False} \\ \textbf{True}: & \textbf{False} \\ \hline \end{array} \\ \textbf{True}: & \begin{array}{|ll|} \hline \text{input} & \text{output} \\ \hline \textbf{False}: & \textbf{False} \\ \textbf{True}: & \textbf{True} \\ \hline \end{array} \\ \hline \end{array}$$

Note that in any world where the conjunction is true, both conjuncts must be true.

We can therefore entail either conjunct from a conjunction premise.

This is called **conjunction elimination**.

Recall the truth table for If:

$$[\![\mathsf{If}]\!]^M = \begin{array}{|ll|} \hline \text{input} & \text{output} \\ \hline \textbf{False}: & \begin{array}{|ll|} \hline \text{input} & \text{output} \\ \hline \textbf{False}: & \textbf{True} \\ \textbf{True}: & \textbf{True} \\ \hline \end{array} \\ \textbf{True}: & \begin{array}{|ll|} \hline \text{input} & \text{output} \\ \hline \textbf{False}: & \textbf{False} \\ \textbf{True}: & \textbf{True} \\ \hline \end{array} \\ \hline \end{array}$$

Note that in any world where the implication and first argument are true, the second must be true.

We can therefore entail the second argument from an implication and first argument premise.

This is called **modus ponens**.

Practice 14.1:

Which of the following are valid entailments?

1. *Etna erupts and Wolf erupts*, so *Etna erupts*.
2. *Etna erupts or Wolf erupts*, so *Etna erupts*.
3. *If Etna erupts then Wolf erupts* and *Etna erupts*, so *Wolf erupts*.

14.3 Natural logic [van Benthem, 1986]

Recall we can generalize quantifiers by whether they are cardinal or proportional.

Recall we can further generalize quantifiers by their relation to their threshold number:

$$\begin{array}{rcll}
[\![\mathsf{Count}_{\geq}\ n\ R\ S]\!]^M & = & [\![|R \cap S| \geq n]\!]^M & \text{(e.g. } \mathsf{Some}\ R\ S = \mathsf{Count}_{\geq}\ 1\ R\ S\text{)} \\
[\![\mathsf{Count}_{\leq}\ n\ R\ S]\!]^M & = & [\![|R \cap S| \leq n]\!]^M & \text{(e.g. } \mathsf{None}\ R\ S = \mathsf{Count}_{\leq}\ 0\ R\ S\text{)} \\
[\![\mathsf{Ratio}_{\geq}\ n\ R\ S]\!]^M & = & [\![|R \cap S| \geq n \times |R|]\!]^M & \text{(e.g. } \mathsf{All}\ R\ S = \mathsf{Ratio}_{\geq}\ 1\ R\ S\text{)} \\
[\![\mathsf{Ratio}_{\leq}\ n\ R\ S]\!]^M & = & [\![|R \cap S| \leq n \times |R|]\!]^M & \text{(e.g. } \mathsf{Few}\ R\ S = \mathsf{Ratio}_{\leq}\ .5\ R\ S\text{)}
\end{array}$$

with variants $Q_{=}$, $Q_{<}$, $Q_{>}$ for other comparison operators defined in terms of these:

$$\begin{array}{rcl}
[\![Q_{=}\ n\ R\ S]\!]^M & = & [\![Q_{\leq}\ n\ R\ S \wedge Q_{\geq}\ n\ R\ S]\!]^M \\
[\![Q_{<}\ n\ R\ S]\!]^M & = & [\![Q_{\leq}\ n\ R\ S \wedge \neg Q_{\geq}\ n\ R\ S]\!]^M \\
[\![Q_{>}\ n\ R\ S]\!]^M & = & [\![Q_{\geq}\ n\ R\ S \wedge \neg Q_{\leq}\ n\ R\ S]\!]^M
\end{array}$$

We can now explain some relatively reliable observations about entailment.

'Less-than' quantifiers with less constrained nuclear scopes entail those with more constraints:

(1) a. *Few cities are capitals.*
    b. (entailed by 1a:) *Few cities are coastal capitals.*

(2) a. *Few cities are coastal capitals.*
    b. (**not** entailed by 2a:) *Few cities are capitals.*

These are called **right downward entailing** quantifiers.

This includes negations:

(3) a. *Cities are not capitals.*
    b. entailed by 3a:) *Cities are not coastal capitals.*

Indeed, negations can be modeled as quantifiers:

$$[\![\mathsf{Not}\ \varphi]\!]^M = [\![\mathsf{None}\ (\lambda_{x:e}\ \mathsf{True})\ (\lambda_{x:e}\ \varphi)]\!]^M$$

'Greater-than' quantifiers with more constrained nuclear scopes entail those with less:

(4) a. *Most cities are capitals.*
    b. (**not** entailed by 4a:) *Most cities are coastal capitals.*

(5) a. *Most cities are coastal capitals.*
    b. (entailed by 5a:) *Most cities are capitals.*

These are called **right upward entailing** quantifiers.

'Less-than' cardinal quantifiers with less constrained restrictors entail those with more constraints:

(6) a. *At most two cities are capitals.*
   b. (entailed by 6a:) *At most two coastal cities are capitals.*

(7) a. *No cities are capitals.*
   b. (entailed by 7a:) *No coastal cities are capitals.*

These are called **left downward-entailing** quantifiers.

The inverses of these quantifiers are also left downward entailing:

(8) a. *All but at most two cities are capitals.*
   b. (entailed by 8a:) *All but at most two coastal cities are capitals.*

(9) a. *All cities are capitals.*
   b. (entailed by 9a:) *All coastal cities are capitals.*

That's because their inverses are 'less-than' cardinal quantifiers with negated nuclear scopes:

(10) a. *At most two cities are not capitals.*
   b. (entailed by 10a:) *At most two coastal cities are not capitals.*

This also includes questions:

(11) a. *Which cities are capitals?*
   b. entailed by 11a:) *Which coastal cities are capitals?*

Indeed, questions can be modeled as quantifiers:

$$\begin{array}{l}\mathsf{All}\ (\lambda_{x:e}\ \mathsf{City}\ x \wedge \mathsf{Capital}\ x) \\ \quad (\lambda_{x:e}\ \mathsf{Want}\ (\mathsf{Intension}\ (\mathsf{Believe}\ (\mathsf{Intension}\ (\mathsf{City}\ x \wedge \mathsf{Capital}\ x))\ \mathsf{Speaker}))\ \mathsf{Speaker})\end{array}$$

But not proportional quantifiers:

(12) a. *Few cities are capitals.*
   b. (**not** entailed by 12a:) *Few coastal cities are capitals.*

(13) a. *Most cities are capitals.*
   b. (**not** entailed by 13a:) *Most coastal cities are capitals.*

These are called **left non-entailing** quantifiers.

Finally, 'greater-than' cardinal quantifiers with more constrained restrictors entail those with less:

(14) a. *Some coastal cities are capitals.*
   b. (entailed by 14a:) *Some cities are capitals.*

These are called **left upward-entailing** quantifiers.

Practice 14.2:

Classify the following as right upward or right downward entailing or neither:

1. *at least seven*
2. *at most seven*
3. *exactly seven*
4. *most*

Practice 14.3:

Classify the following as left upward or left downward entailing or neither:

1. *at least seven*
2. *at most seven*
3. *exactly seven*
4. *most*

Practice 14.4:

Which of the following are valid entailments?

1. *Two volcanoes erupted*, so *Two coastal volcanoes erupted*.
2. *Two coastal volcanoes erupted*, so *Two volcanoes erupted*.

14.4 Negative polarity markers

English marks downward-entailing sub-expressions with negative polarity markers like 'any' (cf. 'some'):

(15) a. *Few cities are capitals with any bridges.*
b. *No cities with any bridges are capitals.*

This works for negation too:

(16) *Cities are not capitals with any bridges.*

This works for questions too:

(17) *Which cities with any bridges are capitals?*

Practice 14.5:

Do the following words behave like negative polarity markers?

1. *at all*
2. *usually*

References

[Carnap, 1947] Carnap, R. (1947). *Meaning and Necessity: A Study in Semantics and Modal Logic*. Chicago: University of Chicago Press.

[van Benthem, 1986] van Benthem, J. (1986). Natural logic. In *Essays in Logical Semantics*. Dordrecht, the Netherlands: Kluwer.