Ling 5801: Lecture Notes 16
Linear Algebra

Complex equations like HMM filtering can be represented efficiently using linear algebra.

Contents

16.1 Terms

We can define matrices and vectors as arrays of real numbers:

- s is a **scalar** iff $s \in \mathbb{R}$

  You will often see scalars written as Greek letters, e.g.: γ

- $\mathbf{v}$ is a **vector** iff $\mathbf{v} \in \mathbb{R}^I$

  Scalars in vectors can be identified by one index: say $\mathbf{v} = \begin{bmatrix} 1.8 \\ -3 \end{bmatrix}$ then: $\mathbf{v}_{[2]} = -3$

- $\mathbf{M}$ is a **matrix** iff $\mathbf{M} \in \mathbb{R}^{I \times J}$

  Scalars in matrices can be identified by two indices: say $\mathbf{M} = \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix}$ then: $\mathbf{M}_{[2,1]} = -3$

16.2 Unary Operations

- **transpose**: for all $\mathbf{M} \in \mathbb{R}^{I \times J}$, and all i, j indices to matrix rows and columns,

  $(\mathbf{M}^\top)_{[i,j]} = \mathbf{M}_{[j,i]}$

  For example: $\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix}^\top = \begin{bmatrix} 1.8 & -3 \\ 12 & 40 \end{bmatrix}$

- **diagonal**: for all $\mathbf{v} \in \mathbb{R}^I$, and all i indices to matrix rows and columns,

  $\text{diag}(\mathbf{v})_{[i,j]} = \begin{cases} \mathbf{v}_{[i]} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

  For example: $\text{diag}(\begin{bmatrix} 1.8 \\ -3 \end{bmatrix}) = \begin{bmatrix} 1.8 & 0 \\ 0 & -3 \end{bmatrix}$

- **Kronecker delta**: for all i, j indices to matrix rows,

  $$(\delta_i)_{[j]} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

  For example: $\delta_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

16.3 Binary Operations

- **scalar sum**: for all $s \in \mathbb{R},\ \mathbf{M} \in \mathbb{R}^{I \times J}$, and all i, j indices to matrix rows and columns,

  $(s + \mathbf{M})_{[i,j]} = (\mathbf{M} + s)_{[i,j]} = s + \mathbf{M}_{[i,j]}$

  (commutative)

  For example: $2 + \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} = \begin{bmatrix} 3.8 & 14 \\ -1 & 42 \end{bmatrix}$

- **matrix/vector sum**: for all $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{I \times J}$, with row and column indices i, j,

  $(\mathbf{M} + \mathbf{N})_{[i,j]} = (\mathbf{N} + \mathbf{M})_{[i,j]} = \mathbf{M}_{[i,j]} + \mathbf{N}_{[i,j]}$

  (commutative)

  For example: $\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2.8 & 14 \\ 0 & 44 \end{bmatrix}$

- **scalar product**: for all $s \in \mathbb{R},\ \mathbf{M} \in \mathbb{R}^{I \times J}$, with row and column indices i, j,

  $(s\,\mathbf{M})_{[i,j]} = (\mathbf{M}\,s)_{[i,j]} = s \cdot \mathbf{M}_{[i,j]}$

  (commutative)

  For example: $2 \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} = \begin{bmatrix} 3.6 & 24 \\ -6 & 80 \end{bmatrix}$

- **matrix/vector product**: for all $\mathbf{M} \in \mathbb{R}^{I \times K}, \mathbf{N} \in \mathbb{R}^{K \times J}$, with indices i, j, k,

  $(\mathbf{M}\,\mathbf{N})_{[i,j]} = \sum_k \mathbf{M}_{[i,k]} \cdot \mathbf{N}_{[k,j]}$

  (*not* commutative)

For example:

$$\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \\ 15 & -6 \\ 7 & 18 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} (1.8 \cdot 1) + (12 \cdot 4) & (1.8 \cdot 2) + (12 \cdot 5) & (1.8 \cdot 3) + (12 \cdot 6) \\ (-3 \cdot 1) + (40 \cdot 4) & (-3 \cdot 2) + (40 \cdot 5) & (-3 \cdot 3) + (40 \cdot 6) \\ (15 \cdot 1) + (-6 \cdot 4) & (15 \cdot 2) + (-6 \cdot 5) & (15 \cdot 3) + (-6 \cdot 6) \\ (7 \cdot 1) + (18 \cdot 4) & (7 \cdot 2) + (18 \cdot 5) & (7 \cdot 3) + (18 \cdot 6) \end{bmatrix}$$

$$= \begin{bmatrix} 49.8 & 63.6 & 77.4 \\ 157 & 194 & 231 \\ -9 & 0 & 9 \\ 79 & 104 & 129 \end{bmatrix}$$

Practice: Complete the following:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (\_ \cdot \_) + (\_ \cdot \_) & (\_ \cdot \_) + (\_ \cdot \_) \\ (\_ \cdot \_) + (\_ \cdot \_) & (\_ \cdot \_) + (\_ \cdot \_) \end{bmatrix}$$

There are two special cases of matrix multiplication for vectors:

1. **inner ('dot') product**: for vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^I$,

   $\mathbf{v}^\top \mathbf{u} = \sum_i \mathbf{v}_{[i]} \cdot \mathbf{u}_{[i]}$

   For example:
   $$\begin{bmatrix} 1.8 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1.8 \cdot 1) + (-3 \cdot 2)$$
   $$= -4.2$$

2. **outer product**: for vectors $\mathbf{v} \in \mathbb{R}^I, \mathbf{u} \in \mathbb{R}^J$,

   $(\mathbf{v}\,\mathbf{u}^\top)_{[i,j]} = \mathbf{v}_{[i]} \cdot \mathbf{u}_{[j]}$

   For example:
   $$\begin{bmatrix} 1.8 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1.8 \cdot 1 & 1.8 \cdot 2 & 1.8 \cdot 3 \\ -3 \cdot 1 & -3 \cdot 2 & -3 \cdot 3 \end{bmatrix}$$
   $$= \begin{bmatrix} 1.8 & 3.6 & 5.4 \\ -3 & -6 & -9 \end{bmatrix}$$

- **Kronecker product**: for all $\mathbf{M} \in \mathbb{R}^{I \times J}, \mathbf{N} \in \mathbb{R}^{K \times L}$,

$$\mathbf{M} \otimes \mathbf{N} = \begin{bmatrix} \mathbf{M}_{[1,1]}\,\mathbf{N} & \cdots & \mathbf{M}_{[1,J]}\,\mathbf{N} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{[I,1]}\,\mathbf{N} & \cdots & \mathbf{M}_{[I,J]}\,\mathbf{N} \end{bmatrix}$$

For example:

$$\begin{bmatrix} 1.1 & 4 \\ -3 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1.1 \cdot 1 & 1.1 \cdot 2 & 1.1 \cdot 3 & 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 1.1 \cdot 4 & 1.1 \cdot 5 & 1.1 \cdot 6 & 4 \cdot 4 & 4 \cdot 5 & 4 \cdot 6 \\ -3 \cdot 1 & -3 \cdot 2 & -3 \cdot 3 & 1 \cdot 1 & 1 \cdot 2 & 1 \cdot 3 \\ -3 \cdot 4 & -3 \cdot 5 & -3 \cdot 6 & 1 \cdot 4 & 1 \cdot 5 & 1 \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1 & 2.2 & 3.3 & 4 & 8 & 12 \\ 4.4 & 5.5 & 6.6 & 16 & 20 & 24 \\ -3 & -6 & -9 & 1 & 2 & 3 \\ -12 & -15 & -18 & 4 & 5 & 6 \end{bmatrix}$$

16.4 Example: Hidden Markov Models

Hidden Markov Model filtering can be represented as a matrix chains:

$$\mathbf{p}^\top = \begin{array}{c} \begin{array}{cc} \text{front} & \text{back} \end{array} \\ \begin{pmatrix} .5 & .5 \end{pmatrix} \end{array}$$

$$\mathbf{A} = \begin{array}{c|cc} & \text{front} & \text{back} \\ \hline \text{front} & .8 & .2 \\ \text{back} & .2 & .8 \end{array}$$

$$\mathbf{B} = \begin{array}{c|ccccccc} & .5\text{kHZ} & .75\text{kHZ} & 1\text{kHZ} & 1.25\text{kHZ} & 1.5\text{kHZ} & 1.75\text{kHZ} & 2\text{kHZ} \\ \hline \text{front} & 0 & 0 & .1 & .2 & .4 & .2 & .1 \\ \text{back} & .1 & .2 & .4 & .2 & .1 & 0 & 0 \end{array}$$

$$\mathsf{P}(Y_2, x_{0..2}) = \mathbf{p}^\top \text{diag}(\mathbf{B}\ \delta_{x_0})\ \mathbf{A}\ \text{diag}(\mathbf{B}\ \delta_{x_1})\ \mathbf{A}\ \text{diag}(\mathbf{B}\ \delta_{x_2})$$

16.5 Vector Normalization

We can normalize these vectors using an n**-norm** of a vector $\mathbf{v}$:

$$\|\mathbf{v}\|_n = \left(\sum_j (\mathbf{v}_{[j]})^n \right)^{\frac{1}{n}} \tag{1}$$

There are several useful instantiations of this:

- The two-norm calculates the length of vector $\mathbf{v}$ as Euclidean coordinates:

$$\|\mathbf{v}\|_2 = \left(\sum_j (\mathbf{v}_{[j]})^2\right)^{\frac{1}{2}} \tag{2}$$

$$= \left(\sum_j \mathbf{v}_{[j]} \cdot \mathbf{v}_{[j]}\right)^{\frac{1}{2}} \tag{3}$$

$$= \sqrt{\textstyle\sum_j \mathbf{v}_{[j]} \cdot \mathbf{v}_{[j]}} \tag{4}$$

For example: $\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2 = \sqrt{2} \qquad \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|_2 = \sqrt{3}$

- The one-norm calculates 'Manhattan distance' (a sum over vector cells):

$$\|\mathbf{v}\|_1 = \left(\sum_j (\mathbf{v}_{[j]})^1\right)^{\frac{1}{1}} \tag{5}$$

$$= \sum_j \mathbf{v}_{[j]} \tag{6}$$

For example: $\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_1 = 2 \qquad \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|_1 = 3$

- The infinity-('inf'-)norm calculates the maximum over vector cells (largest cell dominates):

$$\|\mathbf{v}\|_\infty = \left(\sum_j (\mathbf{v}_{[j]})^\infty\right)^{\frac{1}{\infty}} \tag{7}$$

$$= \max_j \mathbf{v}_{[j]} \tag{8}$$

For example: $\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty = 1 \qquad \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|_\infty = 1$

Norms are useful, as the name suggests, for **normalizing** vectors (resizing them to unit length):

$$\frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

16.6 Cosine Similarity

The dot product of two vectors, after being normalized, is the coordinate of one projected orthogonally onto a (basis) axis defined by the other. The cosine is then the length of this projection (the 'adjacent edge') over one (the 'hypotenuse'):

$$\cos(\mathbf{v}, \mathbf{u}) = \frac{\mathbf{v}^\top}{||\mathbf{v}||_2} \frac{\mathbf{u}}{||\mathbf{u}||_2}$$

This makes a good similarity metric: it's one if $\mathbf{v}$ and $\mathbf{u}$ are aligned, zero if orthogonal:

$$\cos(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = 1$$

$$\cos(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix} = 0$$

Practice:

Recalling that 3,4,5 and 5,12,13 are right triangles,

1. what is the cosine similarity of vectors $\begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}5\\12\end{bmatrix}$, and
2. what is the cosine similarity of vectors $\begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}12\\5\end{bmatrix}$?