Ling 5801: Lecture Notes 13
From Probability Models to Sequence Models

Contents

13.1 Repeated trials

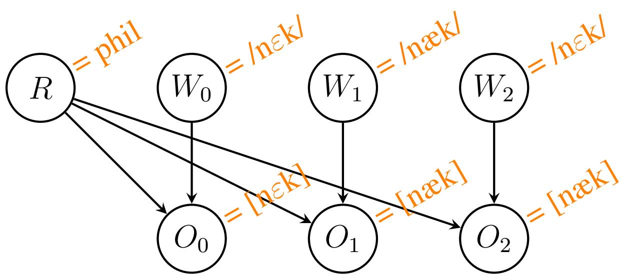
We can extend probability models to include unbounded number of trials.

This is well defined if models are re-used. This is called ***stationarity***.

For example:

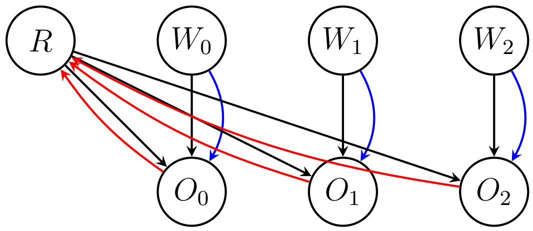
$$\begin{aligned} M_{Pron} = \langle\, & \langle R, W_0, O_0 \rangle, \\ & \{ \langle R, \{\text{ohio}, \text{phil}\} \rangle, \\ & \ \langle W_0, \{/\text{n}\varepsilon\text{k}/, /\text{næk}/\} \rangle, \langle W_1, \{/\text{n}\varepsilon\text{k}/, /\text{næk}/\} \rangle, \ldots, \\ & \ \langle O_0, \{[\text{n}\varepsilon\text{k}], [\text{næk}]\} \rangle, \langle O_1, \{[\text{n}\varepsilon\text{k}], [\text{næk}]\} \rangle, \ldots \}, \\ & \{ \langle R, \emptyset \rangle, \langle W_0, \emptyset \rangle, \langle W_1, \emptyset \rangle \ldots, \langle O_0, \{R, W_0\} \rangle, \langle O_1, \{R, W_1\} \rangle, \ldots \}, \\ & \{ \langle R, \theta_R \rangle, \langle W_0, \theta_W \rangle, \langle W_1, \theta_W \rangle, \ldots, \langle O_0, \theta_O \rangle, \langle O_1, \theta_O \rangle, \ldots \} \,\rangle \end{aligned}$$

Graphically (with example values diagonally):



Inference:

$$\begin{aligned} \mathsf{P}_{\theta_{Wag}}(r) &= \sum_{w_0,w_1,w_2,o_0,o_1,o_2} \mathsf{P}_{\theta_{Wag}}(r, w_0, w_1, w_2, o_0, o_1, o_2) \\ &\overset{\text{def}}{=} \mathsf{P}(r) \cdot \Bigg(\sum_{o_0,w_0} \mathsf{P}(o_0 \mid r, w_0) \cdot \Big(\mathsf{P}(w_0) \Big)_{w_0} \Bigg)_r \cdot \Bigg(\sum_{o_1,w_1} \mathsf{P}(o_1 \mid r, w_1) \cdot \Big(\mathsf{P}(w_1) \Big)_{w_1} \Bigg)_r \cdot \Bigg(\sum_{o_2,w_2} \mathsf{P}(o_2 \mid r, w_2) \cdot \Big(\mathsf{P}(w_2) \Big)_{w_2} \Bigg)_r \end{aligned}$$



Complexity:

$\mathcal{O}(n \cdot |\mathcal{D}_R| \cdot |\mathcal{D}_W| \cdot |\mathcal{D}_O|)$

13.2 Interdependence

Models with unbounded number of variables can have variables be interdependent.

This kind of model is still stationary.

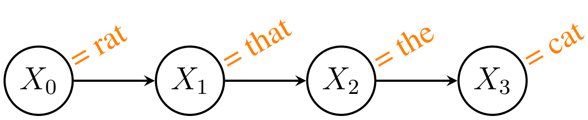
For example:

1. ***Markov chain***: a simple sequence of sucessively dependent variables:

$$\begin{aligned} M_{MC} = \langle\, & \langle X_0, X_1, X_2, \ldots \rangle, \\ & \{ \langle X_0, \mathcal{D}_X \rangle, \langle X_1, \mathcal{D}_X \rangle, \langle X_2, \mathcal{D}_X \rangle, \ldots \}, \\ & \{ \langle X_0, \emptyset \rangle, \langle X_1, \{X_0\} \rangle \langle X_2, \{X_1\} \rangle \ldots \}, \\ & \{ \langle X_0, \pi_X \rangle, \langle X_1, \theta_X \rangle, \langle X_2, \theta_X \rangle, \ldots \} \,\rangle \end{aligned}$$

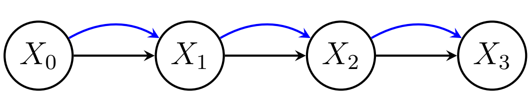
   Note π_X for first variable, θ_X subsequently.

   Graphically:



   Inference:

$$\begin{aligned} \mathsf{P}_{\theta_{MC}}(x_3) &= \sum_{x_0,x_1,x_2} \mathsf{P}_{\theta_{MC}}(x_0, x_1, x_2, x_3) \\ &\overset{\text{def}}{=} \sum_{x_2} \Bigg(_{x_2} \sum_{x_1} \Bigg(_{x_1} \sum_{x_0} \Bigg(_{x_0} \mathsf{P}(x_0) \Bigg) \cdot \mathsf{P}(x_1 \,|\, x_0) \Bigg) \cdot \mathsf{P}(x_2 \,|\, x_1) \Bigg) \cdot \mathsf{P}(x_3 \,|\, x_2) \end{aligned}$$



   Complexity:

   $\mathcal{O}(n)$ (if X observed)

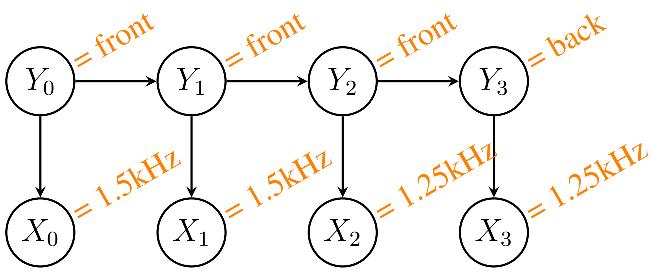
2. ***Hidden Markov model***:

seq. of successively dependent hidden variables ea. w. dependent observation:

$$\begin{aligned} M_{HMM} = \langle\, & \langle Y_0, Y_1, Y_2, \ldots, X_0, X_1, X_2, \ldots \rangle, \\ & \{ \langle Y_0, \mathcal{D}_Y \rangle, \langle Y_1, \mathcal{D}_Y \rangle, \langle Y_2, \mathcal{D}_Y \rangle, \ldots, \langle X_0, \mathcal{D}_X \rangle, \langle X_1, \mathcal{D}_X \rangle, \langle X_2, \mathcal{D}_X \rangle, \ldots \}, \\ & \{ \langle Y_0, \emptyset \rangle, \langle Y_1, \{Y_0\} \rangle \langle Y_2, \{Y_1\} \rangle \ldots, \langle X_0, \{Y_0\} \rangle, \langle X_1, \{Y_1\} \rangle, \langle X_2, \{Y_2\} \rangle, \ldots \}, \\ & \{ \langle Y_0, \pi_Y \rangle, \langle Y_1, \theta_Y \rangle, \langle Y_2, \theta_Y \rangle, \ldots, \langle X_0, \theta_X \rangle, \langle X_1, \theta_X \rangle, \langle X_2, \theta_X \rangle, \ldots \} \,\rangle \end{aligned}$$

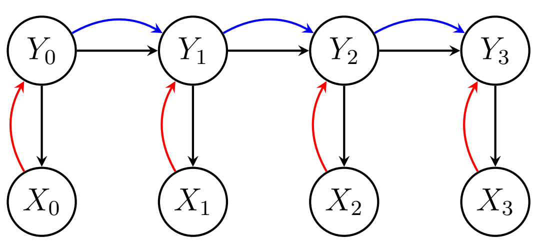
Note π_Y for first variable, θ_Y subsequently.

Graphically:



Inference:

$$\begin{aligned} \mathsf{P}_{\theta_{HMM}}(y_3) &= \sum_{x_0,x_1,x_2,x_3,y_0,y_1,y_2} \mathsf{P}_{\theta_{HMM}}(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) \\ &\stackrel{\text{def}}{=} \sum_{y_2} \Bigg(_{y_2} \sum_{y_1} \Bigg(_{y_1} \sum_{y_0} \Bigg(_{y_0} \mathsf{P}(y_0) \cdot \Bigg(_{y_0} \sum_{x_0} \mathsf{P}(x_0 \,|\, y_0) \Bigg) \Bigg) \cdot \mathsf{P}(y_1 \,|\, y_0) \cdot \Bigg(_{y_1} \sum_{x_1} \mathsf{P}(x_1 \,|\, y_1) \Bigg) \Bigg) \cdot \\ &\qquad \mathsf{P}(y_2 \,|\, y_1) \cdot \Bigg(_{y_2} \sum_{x_2} \mathsf{P}(x_2 \,|\, y_2) \Bigg) \Bigg) \cdot \mathsf{P}(y_3 \,|\, y_2) \cdot \Bigg(_{y_3} \sum_{x_3} \mathsf{P}(x_3 \,|\, y_3) \Bigg) \end{aligned}$$



Algorithm:

\# initialize table of possible states at each time step using start states
for y_0 in Y:
  $V[0, y_0] = \mathsf{P}_{\pi_Y}(y_0) \cdot \mathsf{P}_{\theta_X}(x_0 \,|\, y_0)$

\# for each possible state $y_{t\text{-}1}$ in V at time t, for each $y_{t\text{-}1}, x_{t-1}, y_t$ in M, add y_t
for each t in $1..T$:
  for each $y_{t\text{-}1}$ in Y:
    for each y_t in Y:
      $V[t, y_t] = V[t, y_t] + (V[t{-}1, y_{t\text{-}1}] \cdot \mathsf{P}_{\theta_Y}(y_t \,|\, y_{t\text{-}1}) \cdot \mathsf{P}_{\theta_X}(x_t \,|\, y_t))$

Complexity:

$\mathcal{O}(n \cdot |\mathcal{D}_Y| \cdot |\mathcal{D}_Y|)$ (if X observed)

13.3 Example filtering (estimation of last hidden variable)

For example, in an HMM for estimating front/back tongue position with parameters:

$\pi_Y =$

front	back
.5	.5

$\theta_Y =$

Y_{t-1}	front	back
front	.8	.2
back	.2	.8

(encodes inertia in tongue position)

$\theta_X =$

Y_t	.5kHZ	.75kHZ	1kHz	1.25kHz	1.5kHz	1.75kHz	2kHz
front	0	0	.1	.2	.4	.2	.1
back	.1	.2	.4	.2	.1	0	0

we obtain forward messages calculating $\mathsf{P}(Y_t, X_{1..t})$, marginalizing out all $Y_{1..t-1}$:

t	X_t	$\mathsf{P}(Y_t = \text{front}, X_{1..t})$	$\mathsf{P}(Y_t = \text{back}, X_{1..t})$
0	1.5kHz	$.5 \cdot .4 = .2$	$.5 \cdot .1 = .05$
1	1.5kHz	$.2 \cdot .8 \cdot .4 + .05 \cdot .2 \cdot .4 = .068$	$.2 \cdot .2 \cdot .1 + .05 \cdot .8 \cdot .1 = .008$
2	1.25kHz	$.068 \cdot .8 \cdot .2 + .008 \cdot .2 \cdot .2 = .0112$	$.068 \cdot .2 \cdot .2 + .008 \cdot .8 \cdot .2 = .004$

13.4 Most likely sequence estimation

Subst semiring: $\langle \mathbb{R}_0^\infty, +, \cdot, 0, 1\rangle$ to $\langle \mathbb{R}_0^\infty \times (\mathcal{D}_Y \times \mathcal{D}_X)^*, \text{max-argmax}, \text{prod-pair}, \langle 0, \langle\rangle\rangle, \langle 1, \langle\rangle\rangle\rangle$

where:

$$p, x \text{ max-argmax } q, y = \begin{cases} \text{if } p > q & : p, x \\ \text{otherwise} & : q, y \end{cases}$$

p, x prod-pair $q, y = p \cdot q, \langle x, y\rangle$

Algorithm:

```
# initialize table of possible states at each time step using start states
for y_0 in Y:
  V[0, y_0] = P_{π_Y}(y_0), y_0 prod-pair P_{θ_X}(x_0 | y_0), x_0

# for each possible state y_{t-1} in V at time t, for each y_{t-1}, x_{t-1}, y_t in M, add y_t
for each t in 1..T:
  for each y_{t-1} in Y:
```

for each y_t in Y:
  $V[t, y_t] = V[t, y_t]$ max-argmax $(V[t-1, y_{t-1}]$ prod-pair $\mathsf{P}_{\theta_Y}(y_t \mid y_{t-1}), y_t$ prod-pair $\mathsf{P}_{\theta_X}(x_t \mid y_t), x_t)$

Complexity:

$\mathcal{O}(n \cdot |\mathcal{D}_Y| \cdot |\mathcal{D}_Y|)$ (if X observed)

Example: obtain Viterbi MLS messages w. maximal $\mathsf{P}(y_{1..t}, x_{1..t})$ ending at each y_t:

t	X_t	Y_t = front	Y_t = back
0	1.5	$.5 \cdot .4 = .2$, ⟨front,1.5⟩	$.5 \cdot .1 = .05$, ⟨back,1.5⟩
1	1.5	$\max(.2 \cdot .8 \cdot .4, .05 \cdot .2 \cdot .4) = .064$, ⟨⟨front,1.5⟩, ⟨front,1.5⟩⟩	$\max(.2 \cdot .2 \cdot .1, .05 \cdot .8 \cdot .1) = .004$, ⟨⟨back,1.5⟩, ⟨back,1.5⟩⟩
2	1.25	$\max(.064 \cdot .8 \cdot .2, .004 \cdot .2 \cdot .2) = .01024$, ⟨⟨⟨front,1.5⟩, ⟨front,1.5⟩⟩, ⟨front,1.25⟩⟩	$\max(.064 \cdot .2 \cdot .2, .004 \cdot .8 \cdot .2) = .00256$, ⟨⟨⟨front,1.5⟩, ⟨front,1.5⟩⟩, ⟨back,1.25⟩⟩

13.5 Weighted finite-state automaton

1. Semiring substitution from $\langle \{T, F\}, \vee, \wedge, F, T \rangle$ to $\langle \mathbb{R}_0^\infty, +, \cdot, 0, 1 \rangle$ allows weighted FSA:

   # initialize table of possible states at each time step using start states
   for each q_t in Q:
     $V[0, q_t] = S[q_t]$

   # for each possible state t_{t-1} in V at time t, for each q_{t-1}, x_{t-1}, q_t in M, add q_t
   for each t in $1..T$:
     for each q_{t-1} in Q:
       for each q_t in Q:
         $V[t, q_t] = V[t, q_t] + (V[t-1, q_{t-1}] \cdot M[q_{t-1}, x_{t-1}, q_t])$

2. Now replacing S with probabilities π_Q and factoring M into θ_X and θ_Q gives:

   # initialize table of possible states at each time step using start states
   for q_t in Q:
     $V[0, q_t] = \mathsf{P}_{\pi_Q}(q_t)$

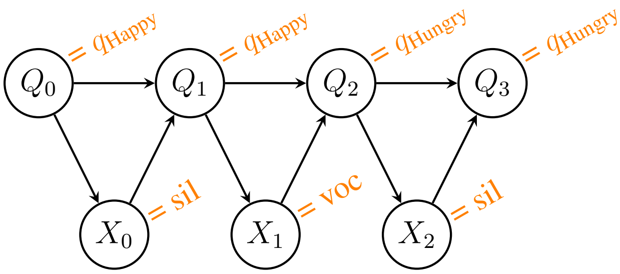
   # for each possible state q_{t-1} in V at time t, for each q_{t-1}, x_{t-1}, q_t in M, add q_t
   for each t in $1..T$:
     for each q_{t-1} in Q:
       for each q_t in Q:
         $V[t, q_t] = V[t, q_t] + (V[t-1, q_{t-1}] \cdot \mathsf{P}_{\theta_X}(x_{t-1} \mid q_{t-1}) \cdot \mathsf{P}_{\theta_Q}(q_t \mid q_{t-1}, x_{t-1}))$

   where: $\mathsf{P}_{\theta_X}(x_{t-1} \mid q_{t-1}) = \sum_{q_t} M[q_{t-1}, x_{t-1}, q_t]$
   $$\mathsf{P}_{\theta_Q}(q_t \mid q_{t-1}, x_{t-1}) = \frac{M[q_{t-1}, x_{t-1}, q_t]}{\sum_{q_t} M[q_{t-1}, x_{t-1}, q_t]}$$

3. This recognizer can be expressed as a probability model:

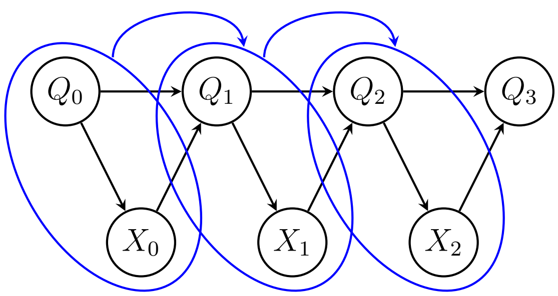
$$\begin{aligned} M_{WFSA} = \langle\, & \langle Q_0, Q_1, Q_2, \ldots, X_0, X_1, X_2, \ldots \rangle, \\ & \{\langle Q_0, \mathcal{D}_Q\rangle, \langle Q_1, \mathcal{D}_Q\rangle, \langle Q_2, \mathcal{D}_Q\rangle, \ldots, \langle X_0, \mathcal{D}_X\rangle, \langle X_1, \mathcal{D}_X\rangle, \langle X_2, \mathcal{D}_X\rangle, \ldots \}, \\ & \{\langle Q_0, \emptyset\rangle, \langle Q_1, \{Q_0, X_0\}\rangle \langle Q_2, \{Q_1, X_1\}\rangle \ldots, \langle X_0, \{Q_0\}\rangle, \langle X_1, \{Q_1\}\rangle, \langle X_2, \{Q_2\}\rangle, \ldots \}, \\ & \{\langle Q_0, \pi_Q\rangle, \langle Q_1, \theta_Q\rangle, \langle Q_2, \theta_Q\rangle, \ldots, \langle X_0, \theta_X\rangle, \langle X_1, \theta_X\rangle, \langle X_2, \theta_X\rangle, \ldots \}\,\rangle \end{aligned}$$

Graphically:



Inference:

$$\begin{aligned} \mathsf{P}_{\theta_{WFSA}}(q_3) &= \sum_{x_0,x_1,x_2,q_0,q_1,q_2} \mathsf{P}_{\theta_{WFSA}}(x_0, x_1, x_2, q_0, q_1, q_2, q_3) \\ &\stackrel{\text{def}}{=} \sum_{q_2} \Bigg(\sum_{q_1,x_1} \Bigg(\sum_{q_0,x_0} \Big(\mathsf{P}(q_0) \cdot \mathsf{P}(x_0 \mid q_0) \Big)_{q_0,x_0} \cdot \mathsf{P}(q_1 \mid q_0, x_0) \cdot \mathsf{P}(x_1 \mid q_1) \Bigg)_{q_1,x_1} \cdot \\ &\qquad \mathsf{P}(q_2 \mid q_1, x_1) \cdot \mathsf{P}(x_2 \mid q_2) \Bigg)_{q_2,x_2} \cdot \mathsf{P}(q_3 \mid q_2, x_2) \end{aligned}$$

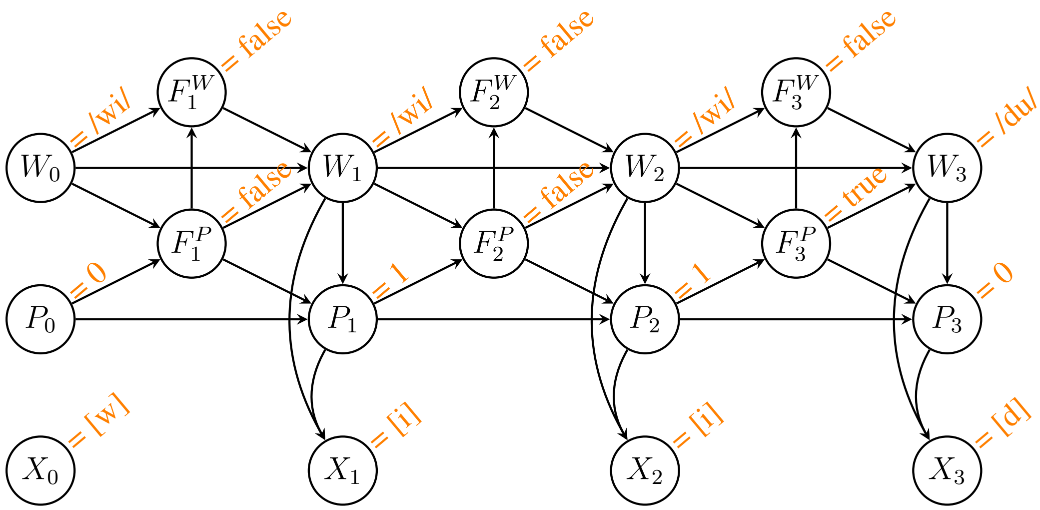


Complexity:

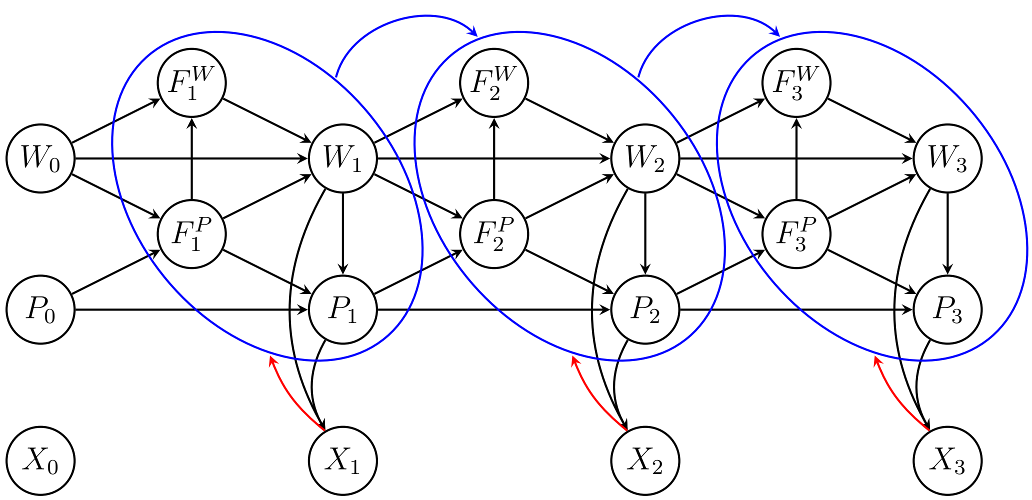
$\mathcal{O}(n \cdot |\mathcal{D}_Q| \cdot |\mathcal{D}_Q|)$ (if X observed)

13.6 Factored sequence model for speech recognition

Synchronize high-/low-level sequences with true/false 'final state' variables:



Inference:



Complexity:

$\mathcal{O}(n \cdot |\mathcal{D}_W| \cdot |\mathcal{D}_P| \cdot |\mathcal{D}_F| \cdot |\mathcal{D}_F| \cdot |\mathcal{D}_W| \cdot |\mathcal{D}_P|)$ (if X observed)

13.7 Weighted pushdown automaton (bounded as hierarchic hidden markov model)

1. Semiring substitution from $\langle \{T, F\}, \vee, \wedge, F, T \rangle$ to $\langle \mathbb{R}_0^\infty, +, \cdot, 0, 1 \rangle$ allows weighted PDA:

   for each time step t:

   for each previous state q_{t-1}^d and store $q_{t-1}^{1..d-1}$ (where $d-1$ is the depth of the store):

   i) for each final state f_t^d and current state q_t^d: (expand +) trans

   $$V[t, q_t^d q_{t-1}^{1..d-1}] = V[t, q_t^d q_{t-1}^{1..d-1}] + (V[t-1, q_{t-1}^d q_{t-1}^{1..d-1}] \cdot M[q_{t-1}^d, \epsilon, x_{t-1}, f_t^d, \epsilon] \\ \cdot M[f_t^d, q_{t-1}^{d-1}, \epsilon, q_t^d, q_{t-1}^{d-1}])$$

ii) for each previous state at deeper level q_{t-1}^{d+1}, final state f_t^{d+1} and current state q_t^{d+1}:

$$V[t, q_t^{d+1} q_t^d q_{t-1}^{1..d-1}] = V[t, q_t^{d+1} q_t^d q_{t-1}^{1..d-1}] + (V[t-1, q_{t-1}^d q_{t-1}^{1..d-1}] \cdot M[q_{t-1}^d, \epsilon, \epsilon, q_{t-1}^{d+1}, q_{t-1}^d] \cdot M[q_{t-1}^{d+1}, \epsilon, x_{t-1}, f_t^{d+1}, \epsilon] \cdot M[f_t^{d+1}, q_{t-1}^d, \epsilon, q_t^{d+1}, q_{t-1}^d])$$

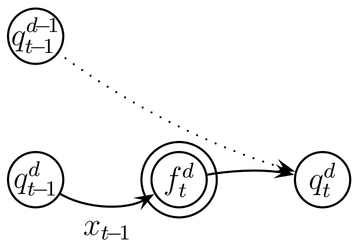
iii) for each final state f_t^d and current state q_t^{d-1}: (expand +) trans + reduce

$$V[t, q_t^{d-1} q_{t-1}^{1..d-2}] = V[t, q_t^{d-1} q_{t-1}^{1..d-2}] + (V[t-1, q_{t-1}^d q_{t-1}^{d-1} q_{t-1}^{1..d-2}] \cdot M[q_{t-1}^d, \epsilon, x_{t-1}, f_t^d, \epsilon] \cdot M[f_t^d, q_{t-1}^{d-1}, \epsilon, q_t^{d-1}, \epsilon])$$

iv) for each previous state at deeper level q_{t-1}^{d+1} and final state f_t^{d+1} and current state q_t^d:

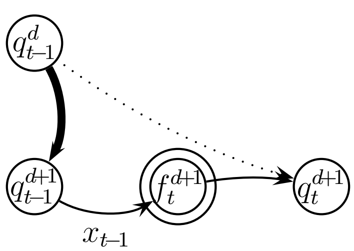
$$V[t, q_t^d q_t^{d-1} q_{t-1}^{1..d-2}] = V[t, q_t^d q_t^{d-1} q_{t-1}^{1..d-2}] + (V[t-1, q_{t-1}^d q_{t-1}^{d-1} q_{t-1}^{1..d-2}] \cdot M[q_{t-1}^d, \epsilon, \epsilon, q_{t-1}^{d+1}, q_{t-1}^d] \cdot M[q_{t-1}^{d+1}, \epsilon, x_{t-1}, f_t^{d+1}, \epsilon] \cdot M[f_t^{d+1}, q_{t-1}^d, \epsilon, q_t^d, \epsilon])$$

i)



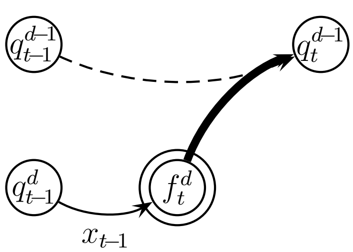
(trans only)

ii)

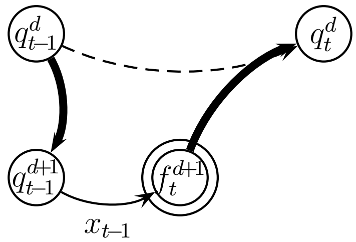


(expand + trans)

iii)



(trans + reduce)

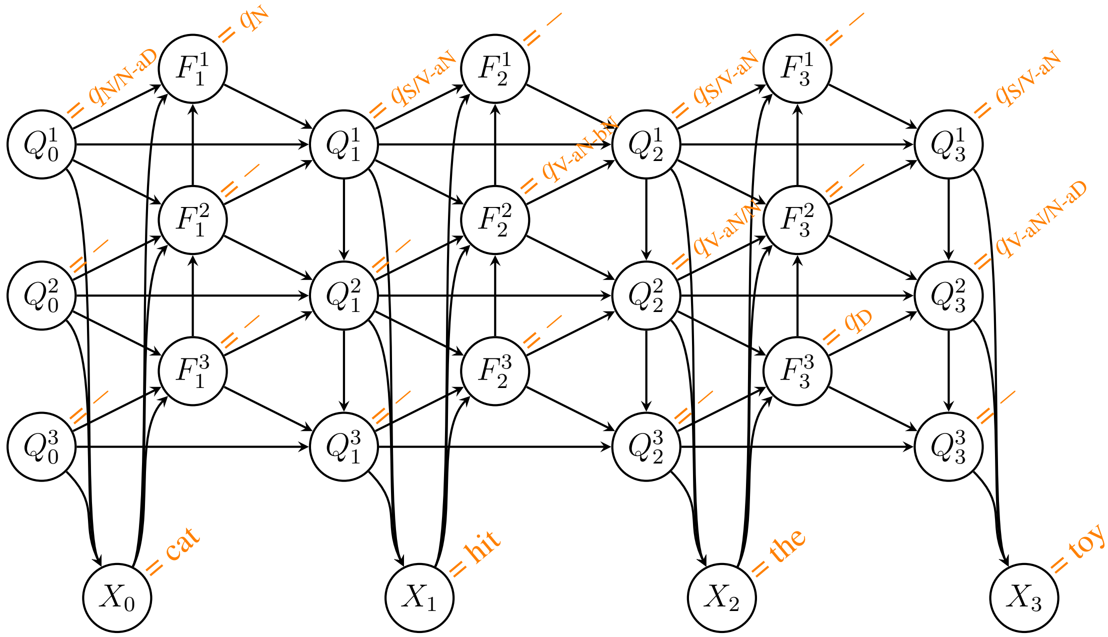
iv)  (expand + trans + reduce)

2. Now replacing S with probabilities π_Q and factoring M into θ_X, θ_F, and θ_Q gives:

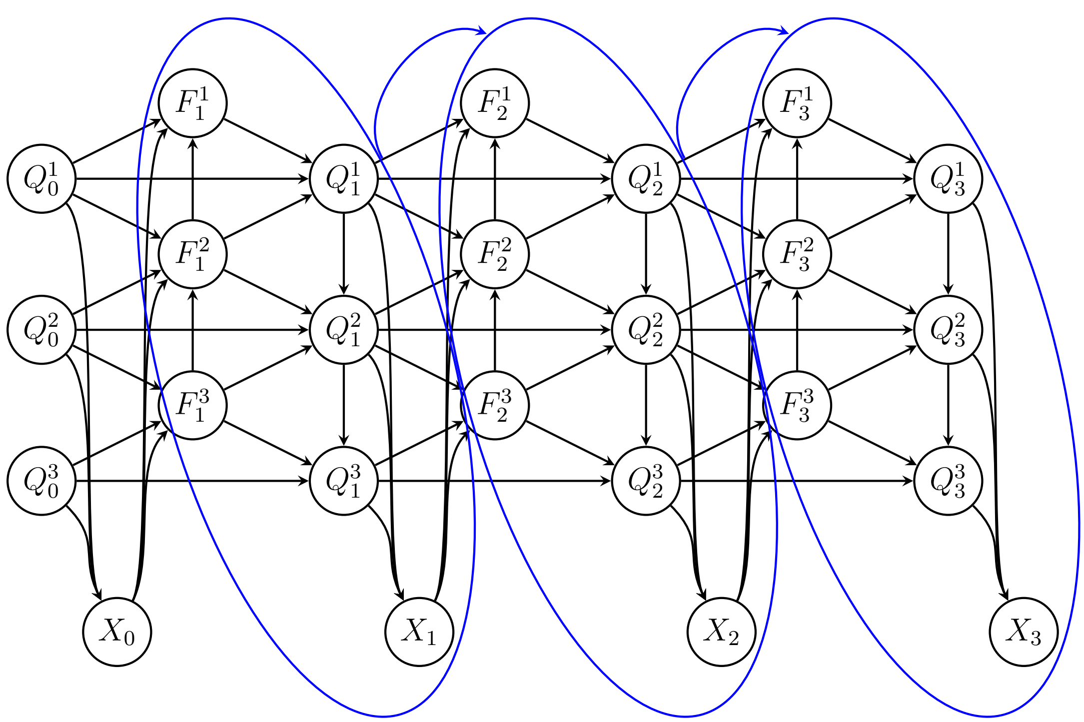
   for each time step t:
   for each previous state q^d_{t-1} and store $q^{1..d-1}_{t-1}$ (where $d-1$ is the depth of the store):
   i) for each final state f^d_t and current state q^d_t: (expand +) trans
   $$V[t, q^d_t q^{1..d-1}_{t-1}] = V[t, q^d_t q^{1..d-1}_{t-1}] + (V[t-1, q^d_{t-1} q^{1..d-1}_{t-1}] \cdot \mathsf{P}_{\theta_X}(x_{t-1} \mid q^d_{t-1}) \cdot \mathsf{P}_{\theta_F}(f^d_t \mid -, \_, q^d_{t-1}, x_{t-1}) \cdot \mathsf{P}_{\theta_Q}(q^d_t \mid f^d_t, -, q^{d-1}_{t-1}, \_))$$
   ii) for each previous state at deeper level q^{d+1}_{t-1}, final state f^{d+1}_t and current state q^{d+1}_t:
   $$V[t, q^{d+1}_t q^d_t q^{1..d-1}_{t-1}] = V[t, q^{d+1}_t q^d_t q^{1..d-1}_{t-1}] + (V[t-1, q^d_{t-1} q^{1..d-1}_{t-1}] \cdot \mathsf{P}_{\theta_X}(x_{t-1} \mid q^d_{t-1}) \cdot \mathsf{P}_{\theta_F}(f^{d+1}_t \mid -, q^d_{t-1}, -, x_{t-1}) \cdot \mathsf{P}_{\theta_Q}(q^{d+1}_t \mid f^{d+1}_t, -, q^d_{t-1}, \_))$$
   iii) for each final state f^d_t and current state q^{d-1}_t: (expand +) trans + reduce
   $$V[t, q^{d-1}_t q^{1..d-2}_{t-1}] = V[t, q^{d-1}_t q^{1..d-2}_{t-1}] + (V[t-1, q^d_{t-1} q^{d-1}_{t-1} q^{1..d-2}_{t-1}] \cdot \mathsf{P}_{\theta_X}(x_{t-1} \mid q^d_{t-1}) \cdot \mathsf{P}_{\theta_F}(f^d_t \mid -, \_, q^d_{t-1}, x_{t-1}) \cdot \mathsf{P}_{\theta_Q}(q^{d-1}_t \mid -, f^d_t, \_, q^{d-1}_{t-1}))$$
   iv) for each previous state at deeper level q^{d+1}_{t-1} and final state f^{d+1}_t and current state q^d_t:
   $$V[t, q^d_t q^{d-1}_t q^{1..d-2}_{t-1}] = V[t, q^d_t q^{d-1}_t q^{1..d-2}_{t-1}] + (V[t-1, q^d_{t-1} q^{d-1}_{t-1} q^{1..d-2}_{t-1}] \cdot \mathsf{P}_{\theta_X}(x_{t-1} \mid q^d_{t-1}) \cdot \mathsf{P}_{\theta_F}(f^{d+1}_t \mid -, q^d_{t-1}, -, x_{t-1}) \cdot \mathsf{P}_{\theta_Q}(q^d_t \mid -, f^{d+1}_t, \_, q^d_{t-1}))$$

   where:
   $$\begin{aligned}
   \mathsf{P}_{\theta_X}(x_{t-1} \mid q^d_{t-1}) &= \textstyle\sum_{f^d_t} M[q^d_{t-1}, \epsilon, x_{t-1}, f^d_t, \epsilon] \\
   &\quad + \textstyle\sum_{f^{d+1}_t, q^{d+1}_{t-1}} M[q^d_{t-1}, \epsilon, \epsilon, q^{d+1}_{t-1}, q^d_{t-1}] \cdot M[q^{d+1}_{t-1}, \epsilon, x_{t-1}, f^{d+1}_t, \epsilon] \\
   \mathsf{P}_{\theta_F}(f^d_t \mid -, _, q^d_{t-1}, x_{t-1}) &= M[q^d_{t-1}, \epsilon, x_{t-1}, f^d_t, \epsilon] \\
   \mathsf{P}_{\theta_F}(f^{d+1}_t \mid -, q^d_{t-1}, -, x_{t-1}) &= \textstyle\sum_{q^{d+1}_{t-1}} M[q^d_{t-1}, \epsilon, \epsilon, q^{d+1}_{t-1}, q^d_{t-1}] \cdot M[q^{d+1}_{t-1}, \epsilon, x_{t-1}, f^{d+1}_t, \epsilon] \\
   \mathsf{P}_{\theta_Q}(q^d_t \mid -, f^{d+1}_t, _, q^d_{t-1}) &= M[f^{d+1}_t, q^d_{t-1}, \epsilon, q^d_{t-1}, \epsilon] \\
   \mathsf{P}_{\theta_Q}(q^d_t \mid f^d_t, -, q^{d-1}_t, _) &= M[f^d_t, q^{d-1}_{t-1}, \epsilon, q^d_t, q^{d-1}_t] \text{ where } q^{d-1}_{t-1} = q^{d-1}_t
   \end{aligned}$$

3. This recognizer can be expressed as a probability model:



Inference:



Complexity:

$\mathcal{O}(n \cdot |\mathcal{D}_Q|^D \cdot |\mathcal{D}_F|^D \cdot |\mathcal{D}_Q|^D)$ (if X observed)

Some observations:

- generates incremental probabilities (good for predicting reading times)

- connects one rule per word (good for efficient/human-like semantic composition)