Assume we have two semi-infinite media with a flat interface that are homogeneous, isotropic, linear, not time varying ($\varepsilon \neq \varepsilon(t)$) and throw away a lot of interesting effects, but covers everyday optics and technology. Work by harmonic solutions in each medium (semi-infinite plane wave).

I got real fields by superposition.

Assume incident, reflected, and transmitted waves.

If we generally match boundary conditions on reflected and incident waves,

$$\vec{E}_i = \vec{E}_i(\vec{n} \cdot \vec{v}_i - \omega t)$$

$$\vec{E}_r = \vec{E}_r(\vec{n} \cdot \vec{v}_r - \omega t)$$

$$\vec{E}_t = \vec{E}_t(\vec{n} \cdot \vec{v}_t - \omega t)$$

Say we match b.c. of $\vec{n}$, i.e.,

At $z=0$ (on $x=0$, $y=0$, $z=0$),

$\vec{E}_i = \vec{E}_r = \vec{E}_t$.

The b.c. must still be satisfied even after the fields, the above must remain in each.

$\vec{n}$ can be independent variables.

(Well, $x$ must be on the interface.)

With no loss of generality, we can make $z=0$ the interface, $\vec{n} = x \hat{x} + y \hat{y}$ is now independent.

Suppose the $\varepsilon$s are different, and phase one equal to $(\vec{0}, \vec{k})$.

Then they can't be equal for $(\vec{0}, \vec{k})$.

So, $\omega_\varepsilon = \omega_n = \omega_k = \omega$.

Red light in gives you red light out.

$U = \frac{\omega}{c}$

$\frac{\varepsilon \omega^2}{c} = \frac{\varepsilon \omega^2}{c} = \frac{\varepsilon \omega^2}{c}$

$\eta = \frac{\omega}{k}$

$\eta_1 \lambda_1 = \eta_2 \lambda_2$.

$\lambda_1 \neq \lambda_2$ for the larger index medium has the shorter wavelength (convention: $\text{SSO on}$ $\lambda_2$, $\text{SSO on}$ $\lambda_1$).
\[ \hat{e}_n \cdot \hat{n} = \hat{r}_n \cdot \hat{n} = \hat{r}_x \cdot \hat{n} \Rightarrow k_x x + k_y y = k_n x \times k_n y = \ldots \]

But, \( x \times y \) are completely independent, so

\[ \begin{align*}
\overline{r}_{x+} &= k_n x x = k_n x \\
k_{x+} &= \hat{r}_x = \hat{k}_x \\
k_{y+} &= \hat{k}_y = \hat{k}_y \\
|k_{x+}| &= |k_x| + |k_y|
\end{align*} \]

With no loss in generality, but \( \hat{r}_i \) in \( x \hat{x} \) plane: all \( y_i = 0 \).

All \( \hat{r}_i \) are in \( x \hat{x} \) plane.

The common plane is the "Plane of incidence"

\[ k_x \hat{x} = k_x \cos(\theta_0 - \phi) = k_x \hat{x} \cos \theta_1 \]

To simplify, we almost always work with \( \phi = 0 \), i.e., only the normal.

The reflected
\[ \hat{x}_n \cdot \hat{n} = \hat{r}_n \cdot \hat{n} \]

Choose \( \Theta_1 \) to be on opposite side of the normal to medium \( \hat{n} \)

\[ k_x \sin \Theta_2 = k_x \sin \Theta_2 \]

\[ \frac{1}{2} \sin \Theta_1 = \frac{1}{2} \sin \Theta_2 \]

\[ \sin \Theta_1 = \alpha \sin \Theta_2 \]

\[ \Theta_2 = \Theta_n = \Theta_1 \]

This is the only way to draw this so that:

(\( \hat{r}_n \) is coplanar in medium \( n \))

(\( \hat{x}_n \) angle \( \phi_1 \))

(\( \hat{n} \), \( \hat{x}_n \), \( \hat{r}_n \))
Here's a picture incorporating what we know for $n_i > n_f$:

\[ \vec{E} = \vec{E}_i \times \vec{E}_f \]
\[ \vec{E}_f = \vec{E}_i \]

These continuity set us the geometrical behavior. To go further, we need MEs.

**General**

\[ \vec{D} \cdot \vec{E} = \frac{\partial \Phi}{\partial x} \]  
\[ \vec{D} \cdot \vec{B} = 0 \]  
\[ \vec{D} \times \vec{E} = -\nabla \Phi \]  
\[ \vec{D} \times \vec{B} = \mu_0 \nabla \times \vec{H} \]

**Specialized for media**

\[ \vec{D} = \varepsilon_0 \vec{E} \]  
\[ \vec{B} = \mu_0 \vec{H} \]

\[ \vec{H} = \mu_0 \vec{B} - \nabla \Phi \]  
\[ \vec{E} = \varepsilon \vec{E} \]

\[ \vec{D} \cdot \vec{E} = 0 \]  
\[ \vec{D} \cdot \vec{B} = 0 \]  
\[ \vec{D} \times \vec{E} = -\nabla \Phi \]  
\[ \vec{D} \times \vec{B} = \mu_0 \varepsilon \nabla \times \vec{E} \]

Last, to get b.c. we switch to integral form. We're not looking at local behavior, but relationships across the interface, so integrals are better. Only need $\vec{E}$, but we'll do all $\vec{H}$. The interface, so integrals over $\ell$.

\[ \oint \vec{E} \cdot d\ell = \nabla \cdot \vec{D} \]  
\[ \text{Note: } A \neq \nabla \Phi \]

- We only care about the interface, so let $\ell$ be small: $\int d\ell = 0$

- For a bottom, don't contribute to $\oint$

- Let $\ell$ be small (not zero) so $\vec{E}$ is constant. Let $\ell$ will do for any incident angle (0° worst case).

Figure based on any figure above.

\[ \vec{E}_i \cdot \vec{n} = \vec{E}_f \cdot \vec{n} \]

Possible, not possible.
$\varepsilon \mathbf{E} \cdot \mathbf{d} \hat{a} = \mu_0 \int \mathbf{J} \cdot \mathbf{d} \hat{a}$ ........ $\mathbf{B}_{1,0} = \mathbf{B}_{2,0}$

Divergence will yield normal components:

$\oint_{S} \mathbf{B} \cdot \mathbf{d} \hat{a} = 0$

$\mathbf{B}_1 \cdot \mathbf{A}_1 = \mathbf{B}_2 \cdot \mathbf{A}_2 = 0$

$\mathbf{B}_1 \cdot \mathbf{A}_1 = \mathbf{B}_2 \cdot \mathbf{A}_2 = 0$ because $\mathbf{A}_1 = -\mathbf{A}_2$

$\mathbf{A}_1 \cdot \mathbf{A}_2 = 0$

$L_s$ will yield $\oint_{S} \mathbf{B} \cdot \mathbf{d} \hat{a} = 0$ \[\Rightarrow\] $\mathbf{B}_{1,0} = \mathbf{B}_{2,0}$

$\varepsilon_1 \mathbf{E}_{1,0} = \varepsilon_2 \mathbf{E}_{2,0}$

Possible \text{ not possible}

So, the normal and tangential be. for $\mathbf{E}$ are different

(Could would be for $\mathbf{B}$ if we used magnetic media - we won't).

Even more different when we treat conductors.

In general, $\mathbf{E} \times \mathbf{B}$ will have both components which have to be treated simultaneously. Instead, let's try a special case where it's easier and use superposition to get the general case.

"P-polar" "S-polar"

"H-polar" "A-polar"

Note: so, of two of the following will define the "plane of incidence":

$\varepsilon$ \text{ \& $\mathbf{E}$} \text{ \& $\mathbf{H}$} \text{ \& $\mathbf{B}$}

\text{interfacial normal}

\text{usually easiest}

1. $\mathbf{E}$ is in plane of incidence
2. $\mathbf{E}$ is pure tangential
3. $\mathbf{E}$ is in plane and mixed
4. $\mathbf{E}$ is in plane of incidence
5. $\mathbf{B}$ is pure tangential
6. $\mathbf{B}$ is in plane and mixed
7. $\mathbf{B}$ is in plane and mixed
If the incident wave is "s", then so are the output waves, likewise for "p." This follows from M.E.S.

I'll just illustrate one case: "s" input, but the transmitted wave acquires a normal component.

\[ E_{s0} = E_{s} E_{s0} \text{ and } E_{r} = E_{r0} + E_{r0} \]

Note: so the reflected wave must also have a normal component.

In any case, clearly \( E_{s} \) and \( E_{r} \) are not perpendicular, not possible!

Fresnel Equations (reflected/transmitted amplitudes as a function of \( E_{0} \))

\[ E_{s0} : E_{0s} \times E_{0s} = E_{0s} \]

\[ B_{r0} : -B_{0s} \cos \theta_{s} + B_{0r} \cos \theta_{r} = -B_{0x} \cos \theta_{x} \]

\[ B = \frac{E}{V} = \frac{n}{c} E \]

\[ n \_c (E_{0s} - E_{0r}) \cos \theta_{x} = \eta \_x E_{0x} \cos \theta_{x} \]
It turns out notation of \( E_{0i} \) are convenient so divide by \( E_{0i} \):

\[
1 + \frac{E_{0r}}{E_{0i}} = \frac{E_{0x}}{E_{0i}} \quad \Rightarrow \quad \frac{E_{0x}}{E_{0i}} - \frac{E_{0r}}{E_{0i}} = 1
\]

\[
\Rightarrow \quad \eta_x \cos \theta_x \frac{E_{0x}}{E_{0i}} + \eta_r \cos \theta_r \frac{E_{0r}}{E_{0i}} = \eta_x \cos \theta_x
\]

Two unknowns: \( \eta_x, \eta_r \)

Solving (see Ex 3.4):

\[
\eta_x = \frac{\eta_r \cos \theta_r - \eta_r \cos \theta_x}{\eta_r \cos \theta_r + \eta_r \cos \theta_x}
\]

\[
\eta_r = \frac{2 \eta_r \cos \theta_r}{\eta_r \cos \theta_r + \eta_r \cos \theta_x}
\]

Given \( E_{0x}, \theta_x, \eta_r \), we can use Snell's Law to get \( E_{0r} \).

Thus, we have \( E_{0r} = E_{0x} \) and of course, the \( \theta_r \).

Note: These are field ratios. We don't have the reflectivity, for example,

\[
\eta_x + \eta_r \neq 1 \quad \text{or even} \quad \eta_x^2 + \eta_r^2 \neq 1
\]

\[
\eta_x \neq \eta_r \quad \text{for} \quad \sin \rightarrow \sin \quad \text{is not the same as} \quad \text{for} \quad \sin \rightarrow \sin.
\]

Example 3.2(a) using \( \eta_x = 1.50 \) and \( \theta_x = 0 \) \( \Rightarrow \) \( \theta_r = 0 \)

\[
\eta_x = \frac{\eta_x - \eta_r}{\eta_x + \eta_r} \quad \eta_r = \frac{2 \eta_x}{\eta_x \eta_r}
\]

\[
\sin (\theta_x = 0) \rightarrow \sin (\theta_x = 1.50) \quad \Rightarrow \quad \sin (\theta_x = 1) \rightarrow \sin (\theta_x = 0)
\]

\[
\eta_x = -0.200 \quad \theta_x = 0.800
\]

"round incidence"

\[
\eta_x = 0.200 \quad \theta_x = 1.20
\]
Why draw \( \mathbf{B} \) on \( \mathbf{E} \)?

We could have used \( \mathbf{E} \). Either will do.

But, we focused on \( \mathbf{E} \), not \( \mathbf{B} \).

This choice keeps the \( \mathbf{E} \)'s pointing with positive trajectory components in our reference frame.

Since the L-core also had this,
we'll get similar sign conventions:
reflecting off a dense medium has \( \eta > 0 \).

\[
\eta \mathbf{E}_{\text{in}} - \eta \mathbf{E}_{\text{out}} = \eta \mathbf{E}_{\text{out}}
\]

\[
E_{\text{in}} \cos \theta_{\text{i}} + E_{\text{out}} \cos \theta_{\text{d}} = E_{\text{out}} \cos \theta_{\text{d}}
\]

\[
\eta \frac{E_{\text{out}}}{E_{\text{in}}} = n_i
\]

\[
\cos \theta_{\text{i}} E_{\text{out}} - \cos \theta_{\text{d}} E_{\text{in}} = \cos \theta_{\text{d}}
\]
\[
\eta = \frac{\eta_i \cos \theta - \eta_x \cos \theta_x}{\eta_i \cos \theta + \eta_x \cos \theta_x}, \quad \text{Note \textquoteleft\textquoteleft mixed indices\textquoteright\textquoteright}.
\]

\[
\xi = \frac{2 \eta_i \cos \theta_x}{\eta_i \cos \theta + \eta_x \cos \theta_x}.
\]

Repeating our example of normal incidence \( \theta_x = 0 \rightarrow \theta_x = 0 \)

\[
\nu = \frac{\eta_i - \eta_x}{\eta_i + \eta_x} = \nu,
\]

\[
\xi = \frac{2 \eta_i}{\eta_i + \eta_x} = \xi
\]

To either case, the electric field interaction at the interface is the same: \( \vec{E} \) is pure tangential.

The result must be the same.

We'll spend a fair amount of time, only trying to understand the behavior predicted by these equations.

Organizational questions: When do they min or max? When do they change sign?

Difference \( \eta_i > \eta_x \) and \( \eta_i < \eta_x \)

Terminology: \( \eta_i < \eta_x \) "going from fast to slow" "external incidence" text was thin a lot think going air \( \rightarrow \) Glass.
Brewster's Angle

If \( \gamma \) can be zero at \( \Theta_1 \) internal or external incidence,

\[
\eta_i = \frac{n_x \cos \Theta_i - n_x \cos \Theta_i}{n_x \cos \Theta_i + n_x \cos \Theta_i} = 0
\]

\( n_x \cos \Theta_i = n_x \cos \Theta_i \) \quad \text{but} \quad \eta_i = n_x \sin \Theta_1 \quad \text{(Snell's Law)}

\[
\eta_i \sin \Theta_i = n_x \cos \Theta_i \cdot \sin \Theta_i
\]

but \( \sin \Theta = 2 \sin \Theta \cos \Theta \)

\[
\sin \Theta_1 = \sin \Theta_1 \quad \Rightarrow \quad 2 \Theta_1 = \Theta_1
\]

\( 2 \Theta_1 = 180^\circ - 2 \Theta_1 \) \quad \text{No}

\( \Theta_1 = \Theta_1 \) \quad \text{only if}

\( 0 \leq \Theta_1 \leq 90^\circ \)

\( \Theta_2 + \Theta_2 = 90^\circ \)

\[
\eta_i \cdot \sin \Theta_2 = n_x \sin(90^\circ - \Theta_2) = n_x \cos \Theta_2 \quad \text{and}
\]

\( \tan \Theta_2 = \frac{n_x}{n} \) \quad \text{p-polarized only}

A surface angled at Brewster's angle will not reflect p-polarized light. No reflection loss!!

\[\text{air to glass:} \quad \tan \Theta_2 = \frac{1.5}{1.0} \quad \Theta_2 = 56.3^\circ \]

angle in glass:

\[
n_{\text{air}} \sin \Theta_{\text{air}} = n_{\text{glass}} \sin \Theta_2
\]

\[
\sin \Theta_2 = \frac{n_{\text{air}} \sin \Theta_{\text{air}}}{n_{\text{glass}}} = \frac{1.0}{1.5} \sin \Theta_{\text{air}}
\]

\( \Theta_{\text{air}} = 33.7^\circ \)

\( \Theta_2 = 33.7^\circ \)
Polarization by reflection

As we'll see, if you're close to \( \theta_0 \), you'll still get a significant effect.

"Polarized light"

Note: "s" and "p" relative to interface, "horizontal" and "vertical" relative to the floor.

Total Internal Reflection (complete reflection for \( \theta > \theta_c \), internal incidence)

\[
\eta_x \sin \theta_x = \eta_z \sin \theta_i
\]

\[
\sin \theta_x = \frac{\eta_i}{\eta_x} \sin \theta_i
\]

Define: \( \theta_c = \theta_0 \) when \( \sin \theta_x = 1 \) or \( \theta_x = 90^\circ \)

I'll just cite results for text:

\[
\begin{align*}
\text{s-polarized} & \Rightarrow \quad E_{\text{out}} = E_{\text{in}} e^{i \phi} \\
\text{p-polarized} & \Rightarrow \quad E_{\text{out}} = E_{\text{in}} e^{i \phi} \\
\text{l-polarized} & \Rightarrow \quad E_{\text{out}} = E_{\text{in}} e^{i \phi}
\end{align*}
\]

Where \( \phi = 2 \pi n_z^{-1} \left( \frac{\eta_x}{\eta_z} \right) \)

For \( \theta > \theta_c \), FIR so the beams for grazing incidence.

There's no transmitted wave, but there are fields in the \( \eta_x \) medium as we'll see later.