Let's have the vector nature of $\mathbf{E}$ for this. We write:

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(x, y, z) \mathbf{e}^{-j\omega t}$$

- Complex magnitude varies in space
- Vector in space
- Propagated along $z$
- Plane wave would have $\mathbf{E} = 1$

The lowest or Gaussian beam is given by:

$$u(x, y, z) = u_0 \left( \frac{x^2 + y^2}{2z} \right) $$

$$n = \sqrt{k \cdot n}$$

$$\omega / c = \omega_0 / c \left( 1 + \frac{z^2}{2z^2} \right)$$

$$z = \frac{\omega_0^2}{\lambda}$$

Cylindrically symmetric

$$R(z) = z \left( 1 + \frac{z^2}{2z^2} \right)$$

$$d(z) = \sqrt{z} \left( \frac{3}{2z} \right)$$

Everyone of these terms has an important and straightforward interpretation.

We'll spend some time discussing this. $u(x, y) = u_0 e^{-x^2/\lambda}$

First note that $e^{-2kz}$ does $e^{\frac{\omega_0^2}{2z^2}}$ just affect the field phase

$$I \propto |E|^2 \propto |u|^2 = \frac{u_0^2}{z^2} e^{-2\omega_0^2 \frac{z^2}{2z^2}}$$

So we must need to look at these terms to figure out the intensity profile.
$Z = 0$

$|U(X, Y, Z)| = \frac{w_0}{w} e^{-\frac{x^2}{w^2}}$

$w_0 = \frac{1}{2} \text{ field width}$

$Z \neq 0$

$|U(X, Y, Z)| = \frac{w_0}{w} e^{-\frac{x^2}{w^2}}$

$\omega_0 > \omega_0$

Intensity $D$ defined by $\exp\left(\frac{-x^2}{w^2}\right) = \frac{1}{e}$

$R_\perp = \frac{-\ln{1/e}}{2}$

$\rho_0 = w_0 \sqrt{\frac{\ln{2}}{2}}$

$D = 2 \pi \rho_0$

$Z_0$ is distance from the spot size to reach $R_\perp w_0$

on the $U(z = 0)$ to reach $\frac{1}{2}$

on the $I(z = 0)$ to reach $\frac{1}{2} Z_0$

$-Z_0 \leq Z \leq Z_0$ is region over which the beam is collimated.

Finally:

A Gaussian beam $I(z) \propto 1 + \frac{1}{1 + \left(\frac{z}{Z_0}\right)^2}$
\[ \lambda = \frac{1}{3} \text{ mm} \]

\[ \omega = \frac{1}{\mu} = 1000 \text{ rad} \quad (D = 1.2 \text{ mm}) \]

\[ z_0 = \frac{\pi \omega^2}{\lambda} \approx 6 \text{ m} \]

\[ \Theta = \frac{\lambda}{4\pi \omega} = \frac{1}{6000} = 0.01^\circ \]

A real beam would do as well.
Now let's look at the case:

Recall: \( E(z) \) is close to a plane wave

\( e^{-ikz} \) see the corrections.

It helps to look at a spherical wave in the paraxial region:

Recall:

\[
E_{xy} \approx \frac{e^{-ikR}}{R}
\]

\( R \) is distance from point source

\( = \) radius of curvature

\( = \) radius of spherical wavefront

\( \sim z + \frac{z^2}{2R} \)

\[ R \approx z + \frac{z^2}{2R} \]

\( e_{xy} \approx \frac{e^{-izk}}{R} \rightarrow \) correction

For our Gaussian:

\( U(z) \propto e^{-ikz^2/2} \)

\[ U(z) \propto e^{-ikz^2} \]

The surfaces of constant phase are spherical!

by radius of curvature \( R \) for a wave continually \( \psi(z) \).

But \( R \neq z \):

\[
R(z) = z \left( 1 + \frac{z^2}{2R} \right)
\]

\( e \) vs \( z \)

\( \frac{1}{2}z^2 \rightarrow R = \infty \)

More accurately:

The lines of constant phase are parabolic

The beam profile is hyperbola.
Finally: \( e^{\alpha(c^2)} = e^{\frac{\alpha}{c^2}} \)

\[ \text{As you go through a focus the plane front scattered by it converges to a plane wave.} \]

\[ \text{A Gaussian does not have a well-defined } \lambda. \]

Only a plane wave does that.

Now if you're in the far field, Gaussians are reasonably well understood using ray optics.

In the near field where laser lines, ray optics gives the wrong answer.

Define \( \frac{1}{\delta} = \frac{1}{(\pi c\sigma)^{2}} - 2 \left( \frac{\delta}{\pi c\sigma^{2}} \right) \)

\( \delta(c\beta) \) characterizes the beam at \( z \).

\[ \text{ABC} \text{D rule for Gaussian beam propagation.} \]

\[ \text{Similar to our rule for spherical piston front,} \]

\[ \text{We can rewrite out Gaussian then as:} \]

\[ u_{y}(y,z) = \frac{\alpha}{2c(2\pi)^{2}} e^{\frac{-\alpha}{2c^2}} e^{i k y} \]
\[ \begin{array}{c}
\text{Transmission in } \sin \\
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}_\sin = \begin{bmatrix}
    1 & 4 \\
    0 & 1
\end{bmatrix}
\end{array} \]

\( E_{out} = E_{in} + L \)  
(Recall for spherical waves)
we found \( R_{out} = R_{in} + L \)

\[ \begin{array}{c}
\text{Lens} \\
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}_{\text{lens}} = \begin{bmatrix}
    1 & 0 \\
    -1/\gamma & 1
\end{bmatrix}
\end{array} \]

\( E_{out} = \frac{E_{in}}{\gamma} + L/\gamma \)

\[ \frac{1}{E_{out}} = \frac{1}{E_{in}} - \frac{1}{\gamma} \sqrt{ \gamma } \]

\( W(\xi) \text{ is undepressed} \)

\[ \frac{1}{\xi_{out}} = \frac{1}{\xi_{in}} - \frac{1}{\gamma} \]
Example (from Varis 880 lecture notes p.24) G 4.6 in our text

Input beam has a waist (\( W_0 \)) on a lens \( f \).
Find location and size of output waist.

\[ \begin{align*}
\text{Input beam (}\,z=2\,) & : \\
\frac{1}{\phi_1} & = \frac{1}{R(z)} - i \frac{\lambda}{\pi n r^2(z)} \\
& = \frac{1}{\phi_0} - i \frac{\lambda}{\pi n W_0^2} \\
\text{Ray path?} & : \begin{cases} 
\phi & \text{at } z_1 = 2 \\
\phi & \text{at } z_2 = 2 
\end{cases} \\
\text{Objective}\ z_1 \rightarrow z_2 \\
\text{Work } W_0\text{ in following as } W_1.
\end{align*} \]

Lens \( z = z_2 \)

\[ \begin{align*}
\frac{1}{\phi_2} & = \frac{1}{\phi_1} - \frac{1}{f} = - \frac{1}{f} - i \frac{\lambda}{\pi n W_0^2} \\
\frac{1}{\phi_3} & = - \frac{1}{f} + \frac{1}{\sqrt{\frac{\lambda}{\pi n W_0^2}}} \\
& = - \frac{a + ib}{a^2 + b^2} \\
\end{align*} \]

Displacement \( z = z_3 \)

\[ \begin{align*}
\phi_3 & = \phi_2 + k \\
& = \phi_2 + k \\
\phi_3 & = \frac{1}{k_3} - \frac{i \lambda}{\pi n W_3^2} \\
& = \left( \frac{-a}{2\pi \cos^2(k) + k} \right) + \frac{ib}{2\pi \cos^2(k) + k} \\
& = \left( \frac{-a}{2\pi \cos^2(k) + k} \right) + \frac{b}{2\pi \cos^2(k) + k} \\
\end{align*} \]

\( k \) get \( k \) for next page
Now, $K_2 = 0$ so $\text{Re} \left( \frac{1}{6_2} \right) = 0$ or

\[
K = \frac{a}{\omega^2 + b^2} = \frac{1}{\omega^2 + \frac{\omega^2}{\omega_0^2}} = \frac{f}{1 + \left( \frac{f}{\lambda} \right)^2}
\]

If $\lambda_0$ is large (input beam is a good approximation to a collimated beam), $K \approx f/\lambda$ (If $\lambda_0$ is small, $K \to 0$).

Look at $\omega_3$

\[
\sqrt{\omega_3} \approx -\frac{1}{f \omega_0^2} \frac{\omega_0^2}{\omega_0^2} = -\frac{a^2 + b^2}{6} = \frac{f^2 + \left( \frac{\lambda_0}{\lambda_0} \right)^2}{6}
\]

Using

\[
\omega_3 = \frac{f^2}{\omega_0^2} \frac{\lambda_0}{\lambda_0} \omega_0 = \frac{f^2}{\omega_0^2} \frac{\lambda_0}{\lambda_0} \omega_0
\]

\[
\frac{\omega_3}{\omega_0} = \frac{f/\lambda_0}{\sqrt{1 + (f/\lambda_0)^2}}
\]

If $\lambda_0 >> f:$

\[
\frac{\omega_3}{\omega_0} \approx \frac{f}{\lambda_0} = \frac{f}{\lambda_0}
\]

$\omega_3 \approx \frac{f}{\lambda_0} = \frac{1}{\pi} \frac{(f)}{(\lambda_0)}$

$= \frac{1}{\pi} \frac{f}{\lambda_0}$

F-number $= \frac{f}{\text{Spot size on lens}}$

$= \frac{f}{\text{Lens diameter}}$
Higher order modes

\[ u(x, y, z) = \frac{1}{4} \frac{x^2}{z} H_n \left( \frac{x^2}{z} \right) H_n \left( \frac{y^2}{z} \right) e^{-i \kappa z} e^{i \mu x z} \]

This is new: Hermite polynomials

\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \]

\[ H_0(x) = 1 \]
\[ H_1(x) = 2x \]
\[ H_2(x) = 4x^2 - 2 \]

\[ E_{2, m=0} (x, y=0) = \begin{cases} \text{Gaussian} & x=0 \\ \text{Pulse} & x=\pm \infty \end{cases} \]

The mode of \( \lambda m \) is called: TEM \( \lambda m \) (Transverse Electro-Magnetic)

See Yariv for details.

\[ \frac{1}{b} = \frac{1}{k \alpha} - i \frac{\kappa}{\pi b^2} \]
\[ \frac{-i \kappa r}{2 k \alpha} - \frac{k l m}{2 \pi b^2} \]
\[ e^{-i \pi z} \]
\[ \text{See } \text{E} \]
and obtain directly from (2.2-14)
\[ \psi_{l,m}(x, y) = E(x, y) e^{i\omega t} = E_0 H_l \left( 2^{\frac{1}{4}} \frac{x}{a} \right) H_m \left( 2^{\frac{1}{4}} \frac{y}{a} \right) \exp \left( -\frac{x^2 + y^2}{a^2} \right) \]  (6.10-4)

where \( H_l \) is the Hermite polynomial of order \( l \). The eigenvalue \( \beta_{l,m} \) is obtained from (2.2-8a) and (6.10-3)
\[ \beta_{l,m} = k \left[ 1 - \frac{2}{k} \sqrt{\frac{\mu_0}{\mu}} (l + m + 1) \right]^{\frac{1}{2}} \]  (6.10-5)