1.(8 pts.)
(a) Consider the free Dirac lagrangian density for massless fermions and argue that this system has global symmetries,

$$\Psi_{\tilde{L}} = P_L \Psi \rightarrow e^{i\alpha_L} \Psi_{\tilde{L}}, \qquad \Psi_{\tilde{R}} = P_R \Psi \rightarrow e^{i\alpha_R} \Psi_{\tilde{R}}$$

where α_L and α_R are independent parameters. $P_{L,R}$ are the usual left-handed and right-handed projectors (see HW #3)

(b) Modify the lagrangian density of part (a) so that the system has a local gauge symmetry involving just the left-handed fermions.

$$\Psi_{\tilde{L}} = P_L \Psi \rightarrow e^{i\alpha_L(x)} \Psi_{\tilde{L}}$$

The right-handed fermions are unaffected by this transformation.

Note:: This is an example of a gauge theory where left-handed and right-handed fermions are treated differently, i.e. have different gauge transformation properties. It turns out that a similar situation occurs in the Standard Model electroweak theory.
Note:: This is NOT QED (which treats left-handed and right-handed fermions on equal footing)

(c) Scalar QED is the theory of complex scalar fields interacting with photons. Consider two types of complex scalar fields ϕ_1 and ϕ_2. ϕ_1 has mass m_1 and charge $-e$ (same as electron) and ϕ_2 has mass m_2 and charge $2e$. What is the gauge invariant lagrangian density for this system, including photons ?

2.(6pts.)
Consider a system involving the following fields
Complex scalar field ϕ for particles of mass M and charge $+e$
Left-handed fermions $\Psi_{\tilde{L}}$ for particles of charge $-e$
Right-handed fermions $\Psi_{\tilde{R}}$ for particles of charge $-2e$
Vector field A^μ

(a) If under a local $U(1)$ gauge transformation $\phi(x)$ transforms as

$$\phi(x) \rightarrow e^{-i\alpha(x)} \phi(x)$$

how do the other fields transform? Recall that it is the relative charges among different particles that is relevant here.

(b) Write down the minimal Lagrangian density which describes this system and is gauge invariant. Can you have a fermion mass term?

(c) Can you come up with an interaction, $\mathcal{L}_{int}$, that couples two fermionic fields and one scalar field (ϕ or ϕ^*), has no derivatives, and that is gauge invariant, Lorentz invariant and hermitian?

3.(2pts.)

For QED we saw that the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ is constructed such that $D_\mu\Psi$ has the same transformation property as Ψ itself.

$$\Psi \to e^{i\alpha(x)}\Psi, \qquad D_\mu\Psi \to e^{i\alpha(x)} D_\mu\Psi$$

For SU(2) nonAbelian gauge theory

$$D_\mu = \partial_\mu - ig\hat{B}_\mu = \partial_\mu - ig\sum_{a=1}^{3} \frac{\sigma^a}{2} B^a_\mu$$

Show that

$$D_\mu\Psi \to V\, D_\mu\Psi$$

where $V(x) = e^{i\vec{\alpha}\cdot\vec{\sigma}/2}$ = SU(2) unitary matrix, and Ψ transforms as $\Psi \to V\,\Psi$.
Recall that the analogue of $A_\nu \to A_\nu - \frac{1}{e}\partial_\nu\alpha$ in QED becomes $\hat{B}_\nu \to V(x)\left[\hat{B}_\nu + \frac{i}{g}\partial_\nu\right] V^\dagger(x)$.
Also a trick using $\partial_\mu(V^\dagger V) = 0$ may come in handy.

4. (4 pts.)

This problem is a review problem from ordinary Quantum Mechanics, designed to remind you of the Schroedinger versus Heisenberg pictures. Results for the simple 1D harmonic oscillator will be useful later on in Quantum Field Theory.

In the Schroedinger picture time dependence is carried by the state vector $|\Psi\rangle_S$ according to the Schroedinger equation.
In the Heisenberg picture states are time independent whereas operators evolve according to

$$i\hbar\frac{d}{dt}\hat{A}_H = [\hat{A}_H, \hat{H}_H] \qquad (1)$$

The relation between operators in the two pictures is given by,

$$\hat{A}_H(t) = e^{i\hat{H}_S t/\hbar}\hat{A}_S e^{-i\hat{H}_S t/\hbar} \qquad (2)$$

We will assume that neither $\hat{H}_S$ nor $\hat{A}_S$ are explicit functions of time and that initial conditions are such that the Schroedinger and Heisenberg operators coincide at $t = 0$.

The well known example of the 1D oscillator has the following Hamiltonian in the Schroedinger picture,

$$\hat{H}_S = \hbar\omega(\hat{a}_S^\dagger\hat{a}_S + \frac{1}{2}), \qquad [\hat{a}_S, \hat{a}_S^\dagger] = 1 \qquad (3)$$

Starting from (2), what are $\hat{a}_H(t)$ and $\hat{a}_H^\dagger(t)$? You may find the following relation useful : $e^{\hat{A}}\hat{B}e^{-\hat{A}} = e^\gamma\hat{B}$ if $[\hat{A}, \hat{B}] = \gamma\,\hat{B}$.

What happens to the commutation relation in eq.(3) in the Heisenberg picture?