An Aggregate Dynamic Stochastic Model for an Air Traffic System

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Abstract— **In this article, an aggregate stochastic model for an Air Traffic System (ATS) is developed. Specifically, using a stochastic description of an ATS based on Poisson processes, we construct a stochastic dynamic model for aircraft counts in regions of an airspace. As an example, the developed model is used to represent Center counts in the United States ATS. We also discuss parameter determination in the model, present some analyses of the model, and evaluate our methodology. Finally, two extensions of the basic model —a hierarchical model that represents aircraft counts in regions of various sizes at multiple time scales, and a model that incorporates stochastic disturbances such as thunderstorms —are described.**

I. Introduction and Motivation

At any time, thousands of aircraft are in flight across the world. For many reasons, including uncertain take-off times and unpredictable weather, the locations and behavior of these aircraft at a given instant in time cannot be exactly predicted in advance. Because of this intrinsic uncertainty in any Air Traffic System (ATS), we believe that a stochastic approach to modeling an ATS is valuable.

In addition to uncertainty, a second hurdle in describing and understanding an ATS is its complexity. In the U.S. ATS, for example, as many as 5000 aircraft may be in the air at once, flying along different routes among several hundred airports. Furthermore, the dynamics of each aircraft may be affected by numerous events, including control directives and weather. Because of the complexity of such an ATS, it is sometimes impractical—and often not useful—to track and predict the location of each aircraft in making global decisions about the management of the system. An aggregate description of the ATS may be more tractable and effective in this situation.

Based on these general motivations, we develop an aggregate dynamic stochastic model for an ATS, in which the numbers of aircraft in regions of the airspace are tracked at discrete time-steps. These aircraft counts change with time in the model because of stochastic flows—in particular, the aircraft in each region move to contiguous regions or leave the system with some probability during each time-step. As an example, we model the numbers of aircraft in each Center in the U.S. ATS at discrete time-steps of one and ten minutes.

Some aspects of the uncertainty in ATS's have been studied in the literature. For example, the distributions of departure, enroute, and arrival delays of aircraft have been characterized [1]. Also, queueing models for the arrival of aircraft at airports have been developed [2] [3]. In particular, the article [2] assumes a Poisson process description for aircraft arriving at an airport, and computes the average delay incurred due to the constraints on the landing aircraft. In [3], a more accurate description of the process of aircraft arriving at an airport is considered, and is used to estimate landing delays. The effects of uncertainties in weather prediction on air traffic flow have also been considered [4]. Yet another area in which uncertainty has been considered is in the modeling of airport surface traffic [5]. One aspect of the airport surface traffic, the departure operation of an airport, has been characterized using a queueing model [6]. A queueing model has also been used to study delay cost optimization at hub airports [7]. Recently, a deterministic aggregate model for an ATS has been developed and analyzed in [8]. Detailed deterministic models for an ATS, which track the location of each aircraft, have been used to study optimal methods for Traffic Flow Management (TFM) [9].

As far as the authors know, a *stochastic* dynamic model for the global behavior of the ATS has not previously been presented in the literature. As in [8], our model dynamically tracks the numbers of aircraft in regions of the airspace. In contrast to [8], however, we use stochastic models for both the flow of aircraft into the airspace, and for the movement of these aircraft between regions. In addition to the general motivation of exploring stochastic and aggregate descriptions for the ATS, we believe that our model has some specific practical uses. Potential applications of the model include the following:

- Our model may allow quantification of uncertainties in predicted aircraft counts (such as Center or Sector counts in the U.S. ATS). The regional aircraft counts predicted by the model will differ from actual counts, both because of intrinsic uncertainties in the ATS and because of the aggregation in the modeling process. The structure of our model allows us to explicitly compute the degree of uncertainty in the predictions, and to evolve these computations dynamically in time.
- Our model may allow rapid calculation of the behavior of an ATS under many different circumstances (e.g., different initial conditions or different flow patterns due to weather events). We expect that our model can be used to rapidly identify scenarios that may lead to violations in Sector or Center capacities.
- The aggregate model may provide a good framework for studying TFM. The effects over time of a TFM restriction can be more easily determined with an aggregate dynamic

model than with a detailed model of an ATS. We are particularly interested in developing algorithms for placing Miles-in-Trail (MIT) and Minutes-in-Trail (MINIT) restrictions in the context of the aggregate model.
As the model is introduced and analyzed in this article, we will occasionally discuss the possible value of the model in achieving these three aims.

II. An Aggregate Stochastic Model for an ATS

In this section, we formulate a stochastic model for aircraft counts in regions of an ATS. First, a Poisson process description of the flow of aircraft in an ATS is presented. We then discuss some difficulties in directly using this detailed description of air traffic flow to analyze the dynamics of the ATS. Motivated by these difficulties, we construct the aggregate stochastic model and consider its relationship with the detailed description.

A. *Poisson Process Description of an ATS*

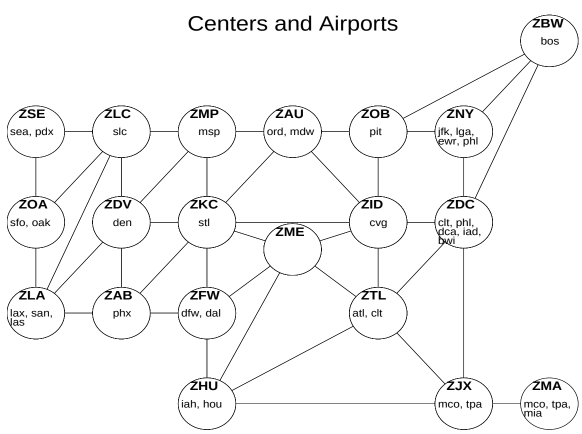

Fig. 1. A network representation of the 20 Centers in the United States. Two Centers are connected in this plot if they are contiguous in the U.S. ATS. Also, the major airports in each Center are listed in smaller font. If an airport is close to the boundary of two Centers, it is listed in both Centers.

The airspace of an ATS is typically subdivided into regions, to facilitate the control and management of aircraft in the airspace. For example, the U.S. airspace is composed of 20 *Centers*, as shown in Figure 1. Aircraft depart from airports distributed among the various Centers, follow routes through the airspace, and arrive at other airports.

Consider an ATS with n Centers. (Although Centers in the U.S. ATS are specific regions of the U.S. airspace, we use the word Center more generally to denote a region of interest in the airspace.) In our stochastic description for the ATS, we assume that the departures of aircraft from each airport are governed by an independent Poisson process with a (possibly) time-varying rate. Later in the article, we will verify from historical data that such a Poisson process description for departures is reasonable.

We also assume that the routes taken by, and the destinations of, the departing aircraft are stochastically independent. This assumption means that the route and destination of a particular departing aircraft does not provide information about the routes and destinations of other departing aircraft. This assumption constitutes an oversimplified representation for the actual flows in the ATS: for example, we might expect that the departing aircraft that are destined for a particular airport roughly follow a periodic schedule, so that the departure of one such aircraft does provide some information about routes and destinations of other aircraft. However, we use the assumption because it allows us to tractably represent flows in the ATS, without worrying about the particular details of departure schedules for aircraft. We also assume that the cruising speed of each aircraft is constant, and that the cruising speed of different departing aircraft are independent.

Now consider the flows of aircraft among Centers in this Poisson process description of an ATS. For notational convenience in this analysis, we introduce a fictitious Center labeled “0”. Aircraft which depart from airports in a Center are said to flow from Center 0 into that Center, while aircraft arriving at an airport in a Center are said to flow from that Center into Center 0. The aircraft flows among Centers in this description can be characterized:

- The aggregate departures of aircraft from all airports in a Center (and their consequent injections into the airspace) are governed by a Poisson process. To see why, note that the aggregate departures in a Center comprise a *merging* of the departures from each airport in the Center, which are each governed by an independent Poisson process. The result of such a merging is well-known to be governed by a Poisson process [10]. We denote the (in general time-varying) rate of the Poisson process governing departures in Center i (or equivalently, flows from Center 0 to Center i) by $\lambda_{0i}(t)$.
- *Aggregate boundary crossings*, or movements of aircraft across a particular boundary from one Center to another one, are also governed by a Poisson process. To see why, first consider the aircraft that depart from a particular airport, fly along a particular route to a destination airport, and have a certain cruising speed. The departures of these aircraft are governed by a Poisson process, since these departures are a *splitting* of all the departures from the airport of interest [10]. Now consider any boundary between two Centers along the route taken by these aircraft. The boundary crossings of these aircraft are also governed by a Poisson process, since each aircraft crosses the boundary after a fixed delay following departure. Finally, the aggregate boundary crossings are the merging of boundary crossings along several routes at several different cruising speeds, each of which are governed by a Poisson process. Thus, the aggregate boundary crossings are governed by a Poisson process. We denote the rate at which aircraft cross a boundary from Center i to Center j at time t by $\lambda_{ij}(t)$.
- Using the same reasoning as for boundary crossings, we find that the aggregate arrivals of aircraft in a Center (i.e., the arrivals of aircraft at all airports in a Center) are governed by a Poisson process. The rate of the aggregate arrivals in Center i (or equivalently, flows from Center i to Center 0) at time t is denoted $\lambda_{i0}(t)$.
- The number of aircraft $s_i(t)$ in Center i at time t is a

Poisson random variable. To see why, again consider the aircraft that depart from a particular airport, fly along a particular route to a destination airport, and have a certain cruising speed, and consider a particular Center along the route traveled by these aircraft. The number of these aircraft that are in the Center of interest at time t is equal to the number that entered the Center between times $t-\widehat{t}$ and t, where $\widehat{t}$ is the (fixed) amount of time needed for the aircraft to pass through the Center. Thus, this number equals the number of boundary crossings into the Center over a time interval $\widehat{t}$. Since these boundary crossings are governed by a Poisson process, the number of these aircraft in the Center is known to be a Poisson random variable. Finally, the total number of aircraft $s_i(t)$ is found by summing Poisson random variables of this sort, and so is Poisson.

Even though we can compute Center count and boundary-crossing statistics using the Poisson process description for the ATS, this description is difficult to use directly for many computations of interest, because aircraft statistics along each particular route must be computed and stored separately. For example, if the total departure rate in a Center is changed in the model (perhaps to reflect the occurrence of inclement weather in that Center), the flows along all routes leaving from each airport in the Center must be recomputed. For similar reasons, the *dynamics* of Center statistics are difficult to determine. For example, let's say that we wish to determine the distribution for the number of aircraft in a Center at some time in the future, given information about the current state of the ATS. The Poisson process description can be used to compute this distribution only if the exact locations of every aircraft in the airspace and the statistics of the possible departures along each route from each airport are explicitly modeled. For applications in which dynamics potentially need to be recomputed for many sets of model parameters, such as control or optimal design applications, such computationally intensive calculations may be infeasible. Thus, we are motivated to develop a simpler aggregate stochastic model for the ATS.

B. *An Aggregate Dynamic Model for Center Counts*

The state variables in our aggregate model are the numbers of aircraft in each Center, tracked at discrete times. Let ΔT be the *time-interval* of the model. Thus, the number of aircraft in each Center is tracked at the times $k\Delta T$, $k = 0, 1, 2, \ldots$ We denote the number of aircraft in Center i at time $k\Delta T$ as $s_i[k]$. Our goal is to develop a model that describes the time-evolution of the state variables $s_i[k]$.

First, between time-steps k and $k+1$, the state variables can change because of aircraft entering each Center upon departure from airports. In our aggregate model, the number of aircraft that depart from airports in Center i, $1 \le i \le n$, between times $k\Delta T$ and $(k+1)\Delta T$ is modeled as a Poisson random variable $U_{0i}[k]$, with mean denoted by $\lambda_{0i}[k]$. In addition to the flows into the ATS due to departures at airports, aircraft may change Centers, or leave a Center through arrival at an airport. In our aggregate model, we envision each aircraft in a Center as moving to another Center or arriving at an airport with some probability during a time-step. In particular, we assume that each aircraft in Center i independently travels to Center j (or leaves the airspace for $j = 0$) between time-steps k and $k+1$ with probability $p_{ij}[k]$. We denote the total number of aircraft that flow from Center i to Center j between times k and $k+1$ by $U_{ij}[k]$. For small enough ΔT, it can be shown that the conditional distribution for the flow $U_{ij}[k]$ given the Center count $s_i[k]$ is well-approximated by a Poisson random variable, with mean $p_{ij}[k]s_i[k]$ [11]. (If ΔT is larger, the $U_{ij}[k]$ must be represented using dependent binomial random variables; the same analyses of the model can be completed in this case, albeit with a little extra computation.) Thus, we have modeled the flows of aircraft among Centers in the airspace, as well as the flows of aircraft arriving at airports.

Now that we have characterized the flows of aircraft in our model, the state variable update can be specified by accounting for the number of aircraft entering and leaving each Center i between times k and $k+1$:

$$s_i[k+1] = s_i[k] - \sum_{j=0,j\neq i}^{n} U_{ij}[k] + \sum_{j=0,j\neq i}^{n} U_{ji}[k], \qquad (1)$$

This update rule defines the temporal evolution of our aggregate stochastic model. The dynamics of the model are depicted pictorially in Figure 2.

In our application of the aggregate model, it is not Equation 1 that we propagate forwards in time. Instead, we propagate expectations and variances of the $s_i[k]$, using equations that are derived from Equation 1, and that have a very simple structure. The details are given in Section 3.2.

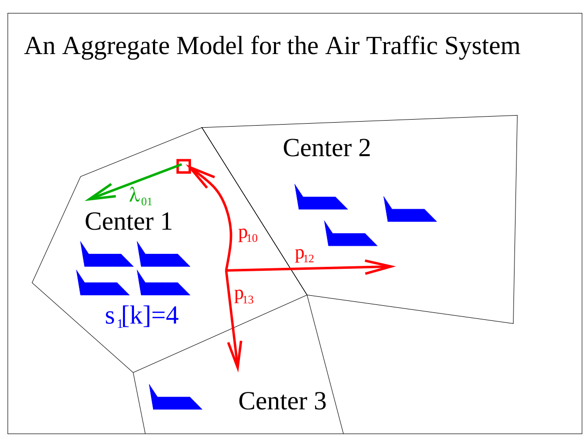

Fig. 2. This figure describes the dynamics of our aggregate stochastic model for the ATS. Aircraft enter Centers according to Poisson processes. Also, during an interval of time, each aircraft in a Center may move to another Center or leave the system, with some probability. We are interested in tracking the number of aircraft in each Center in this model.

Our aggregate model is closely related to the detailed Poisson process description of the ATS discussed in Section 2.1. In the detailed description, the departures of aircraft in each Center i are governed by a Poisson process with rate $\lambda_{0i}(t)$. Thus, between times $k\Delta T$ and $(k+1)\Delta T$, the number of departing aircraft in Center i is a Poisson random variable, with mean $\int_{k\Delta T}^{(k+1)\Delta T} \lambda_{0i}(t)dt \approx \lambda_{0i}(k\Delta T)\Delta T$. If the mean of $U_{0i}[k]$ in the aggregate model is chosen to be $\lambda_{0i}[k] = \lambda_{0i}(k\Delta T)\Delta T$, the statistics of the departures in each time-interval in the aggregate model and detailed description are essentially identical. Unlike departures, the flows among Centers in this aggregate model cannot be made identical to the flows in the detailed description of the ATS (e.g., an aircraft in the aggregate model can flow from Center i to Center j and then return to Center i with some small probability in the aggregate model, while an aircraft in the detailed description follows a route and so would not revisit a Center). However, the probabilities $p_{ij}[k]$ in the aggregate model can be set so that the flows in the aggregate model match the flows in the detailed description in an average sense.

To do so, consider the number of aircraft $s_i[k]$ in Center i at some time-step k in the detailed description of the ATS. It is reasonable to expect that, on average, a certain fraction (possibly 0) of these aircraft will travel to each other Center, or will exit the ATS, during a time interval ΔT. At time-step k (time $k\Delta T$), there are on average $\lambda_{ij}(k\Delta T)\Delta T$, $0 \leq j \leq n$, aircraft that flow to Center j during the next time interval (this includes aircraft exiting the system through arrival at airports, which corresponds to $j = 0$). Furthermore, there are on average $\bar{s}_i(k\Delta T)$ aircraft in Center i. Thus, we might expect that a fraction $\frac{\lambda_{ij}(k\Delta T)}{\bar{s}_i(k\Delta T)}\Delta T$, $0 \leq j \leq n$, of the aircraft in Center i will travel to j between time $k\Delta T$ and $(k+1)\Delta T$. By setting probability $p_{ij}[k]$ in the aggregate model equal to $\frac{\lambda_{ij}(k\Delta T)}{\bar{s}_i(k\Delta T)}\Delta T$, we obtain the same average fraction of aircraft traveling from Center i to Center j as in the detailed model. We can also show (with a little algebra) that the average number of aircraft in Center i, as well as the average number of aircraft that flow from Center i to Center j, are essentially the same for the aggregate and detailed descriptions at each time-step if the $p_{ij}[k]$ are chosen in this way.

III. Parameter Determination, Analysis, and Verification

In this section, we pursue three important questions regarding the stochastic model developed for an ATS:

1. How can the parameters of the model be determined from data?
2. How can the model be analyzed, and why is this analysis useful?
3. Does the model accurately represent some aspects of the behavior of ATS?

Throughout the discussion in this section, examples from the U.S. ATS are used to illustrate our methodology.

A. Parameter Determination

Our aggregate model for Center counts requires three sets of parameters: the time-interval ΔT, the average number of departures $\lambda_{0i}[k]$ in each Center i at time-step k, and the probabilities that aircraft in Center i, $1 \leq i \leq n$, go to Center j, $0 \leq j \leq n$, during time-step k. Because the detailed Poisson process description of an ATS represents the movement of actual aircraft more precisely than the aggregate model, it is more natural for us to estimate parameters of the detailed description from historical data first, and subsequently infer the parameters of the aggregate model. Thus, we focus on estimating the parameters of the detailed description— namely, the mean number of aircraft in each Center ($\bar{s}_i(t)$); and the average departure flow rates, arrival flow rates, and flow rates between Centers ($\lambda_{ij}(t)$). Assuming that the ATS is operating under typical conditions, these parameters can be estimated from historical data. Once these parameters have been found, we can choose a time interval ΔT and approximate the aggregate model's parameters as described in the Section 2, i.e., by setting $\lambda_{0i}[k] = \lambda_{0i}(k\Delta T)\Delta T$ and $p_{ij}[k] = \frac{\lambda_{ij}(k\Delta T)}{\bar{s}_i(k\Delta T)}\Delta T$.

First, consider the mean number of aircraft $\bar{s}_i(t)$ in Center i at time t. In general, we allow this expectation to vary with time in our framework; realistically, we might expect that the mean number of aircraft in each Center would change slowly throughout a single day under typical operating conditions, but would be nearly identical when compared at a certain time over several days. As a simple first attempt at modeling the U.S. ATS, we use a constant value for mean number of aircraft in each Center. These average numbers of aircraft are estimated from actual Center counts during 500 minutes in the afternoon and evening of a particular day, September 6, 2000. For example, we find that, on average, 127.6 aircraft are present in the Seattle Center during this time interval.

The second set of parameters that are necessary for the analysis of the model are the rates $\lambda_{ij}(t)$ of aircraft flow from Center i to Center j (or, for $i = 0$ or $j = 0$, the flow rates into or out of the Center due to departures and arrivals). Like the mean parameters, the rate parameters $\lambda_{ij}(t)$ can be computed by using historical data on the numbers of aircraft that cross each boundary, and that enter into and depart from the system at airports. As with Center count averages, the flow rates across boundaries are expected to vary slowly throughout the course of a day. In our simulations, we have used a crude model in which a single flow rate is estimated based on historical data from September 6th, 2000. For simplicity's sake, these flow rates have been computed assuming that each aircraft flies along the shortest path from its origin to its destination. A more accurate model would require careful measurement of flow rates; here, we are interested in the modeling methodology rather than the accuracy of the specific model, so a more careful computation of the flow rates is not pursued. The average flow rates computed from data and used to construct the model are shown in Figure 3.

Under normal operating circumstances, it is reasonable that mean numbers of aircraft and expected flow rates can be computed from data. Thus, the model can employ these average statistics, combined with actual information on the state of the ATS, to simulate and analyze the future be-

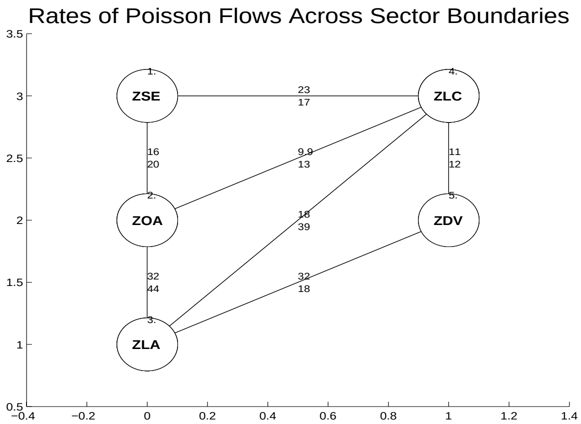

Fig. 3. The expected flow rates between Centers (in aircraft per hour) are shown for five of the Centers in U.S. ATS. These average flow rates were computed using data from September 6, 2000. The top number on each branch show the average number of aircraft moving from the lower-numbered Center to the higher-numbered Center, while the bottom number on each branch shows the average flow in the reverse direction.

havior of the network. However, under unusual conditions (due to bad weather or other disturbances, for example), average behavior most likely cannot be deduced from historical data. In fact, in these aberrant situations, we would hope to compute these averages through the modeling process rather than using them as parameters in the model! In particular, we expect that information about a disturbance can be used to change certain model parameters locally (i.e, near the affected Center), and in turn the model analysis can be used to quantify the behavior of the disturbed ATS.

The final parameter of the model, the time-interval ΔT, should be chosen small enough to capture the fluctuations of interest in the dynamics of the ATS. However, if ΔT is chosen to be too small, unnecessary computation is introduced in the analysis of the model; if ΔT is chosen to be too large, some dynamics of interest may not be modeled, and also the use of a discrete model for the dynamics may introduce significant error. To choose a time-step ΔT for our model for the U.S. ATS, we looked at plots of the average numbers of aircraft in Centers during time intervals of various durations (Figure 4). Based on these plots, we believe that a time-step of less than or equal to 10 minutes is sufficient to capture the dynamics of interest in the ATS.

Once the mean Center counts, expect flow rates, and time-step have been determined, the probability $p_{ij}[k]$ that a randomly chosen aircraft in Center i goes to Center j between $k\Delta T$ and $(k+1)\Delta T$ can be computed for each i and j. Some of these probabilities are shown in Figure 5.

B. Analysis

Once the parameters of our model have been determined from historical data, the model can be analyzed to gain insight into the behavior of the ATS. In particular, given the numbers of aircraft in each Center at the initial time, moments and cross-moments of the numbers of aircraft in each Center at future times can be computed. Thus, we

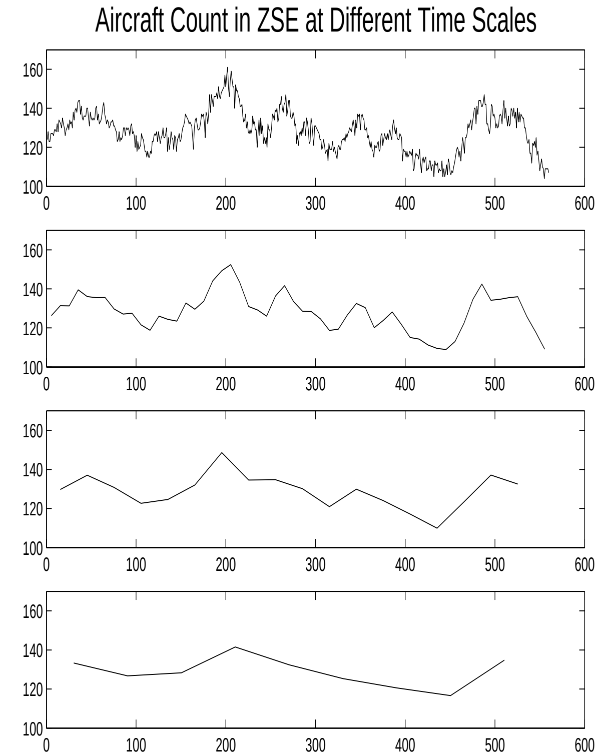

Fig. 4. The time-evolution of the number of aircraft in the Seattle Center (ZSE) during the afternoon and evening of September 6th, 2000, is shown, plotted at four different time scales. In particular, the top plot shows the number of aircraft that were actually present in ZSE, measured at one-minute intervals. The second, third, and fourth plots show the average numbers of aircraft present in ZSE during each ten-minute interval, thirty-minute interval, and one-hour interval, respectively. Ten-minute averages of Center counts accurately approximate the actual Center counts (almost always to within 5 aircraft), suggesting that 10 minutes is a fine enough resolution to capture dynamics of interest in the U.S. ATS.

can predict expected future Center counts and the possible variability in these Center counts using the model. In turn, the expected response and variability in Center counts can be used to identify regions of the airspace that may be prone to excessive traffic. These regions could then be studied more carefully to determine whether or not capacity excesses would actually occur.

To compute the expected numbers of aircraft in each Center at each time-step, given initial conditions, it is helpful to redefine the model specified by Equation 1 in vector notation. Consider the following definitions:

- Define the *state vector* at time-step k to be

$$\mathbf{s}[k] = \begin{bmatrix} s_1[k] \\ \vdots \\ s_n[k] \end{bmatrix}$$

.

- Define the elements of the $n \times n$ *transition matrix* $P[k]$

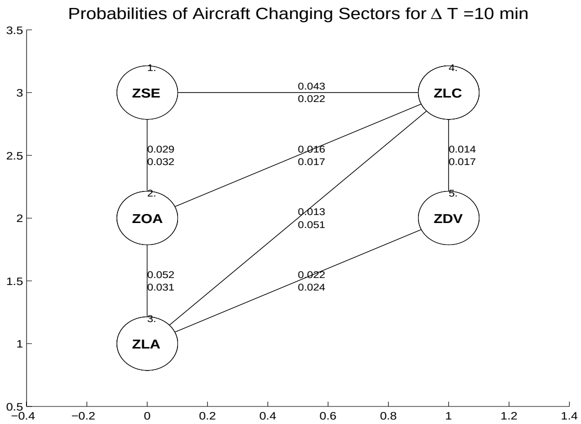

Fig. 5. This plot shows the probability that an aircraft in the aggregate model moves from one Center to another during one time-interval. The upper probability on each branch is the probability that an aircraft from the lower-numbered Center moves to the higher-numbered Center. (In our example, the probabilities do not change with time, since the mean numbers of aircraft and flow rates do not change with time. More generally, however, these probabilities may depend on the time-step k.)

as follows:

$$\text{for } 1 \leq i \leq n, \quad P_{ii}[k] = (1 - \sum_{j=0, j\neq i}^{n} p_{ij}[k])$$

$$\text{for } 1 \leq i \leq n,\, 1 \leq j \leq n, i \neq j, \quad P_{ij}[k] = p_{ji}[k].$$

- Define the *transition vector* to be

$$\beta[k] = \begin{bmatrix} \lambda_{0,1}[k] \\ \vdots \\ \lambda_{0,n}[k] \end{bmatrix}.$$

Our goal is to determine the conditional expectation for the state vector $\mathbf{s}[k]$ given the initial state vector $\mathbf{s}[0]$, or $E(\mathbf{s}[k]|\mathbf{s}[0])$. In fact, these conditional expectations can be found through the following linear recursion:

$$E(\mathbf{s}[1]|\mathbf{s}[0]) = P[0]\mathbf{s}[0] + \beta[0]$$

and

$$E(\mathbf{s}[k+1]|\mathbf{s}[0]) = P[k]E(\mathbf{s}[k]|\mathbf{s}[0]) + \beta[k], \quad k \geq 1. \quad (2)$$

An outline for the derivation of Equation 2 is given in the Appendix. A simulation of the number of aircraft in the Seattle Center ZSE is compared with the expected number (conditioned on the initial Center counts) of aircraft in ZSE in Figure 6. The conditional expected number of aircraft in ZSE depends upon the initial Center counts at the beginning of the simulation, but eventually approaches the steady-state expected number (which is a parameter of the model).

In addition to providing a prediction for the behavior of the ATS, we believe that the recursion for mean Center counts is valuable because it allows us to determine the sensitivity of Center counts to various parameters in the model, including initial conditions and steady-state average flow rates. The sensitivity analysis of the model is not presented in any further detail here, but we note that sensitivities of the expected state vector to parameter changes can be deduced from Equation 2.

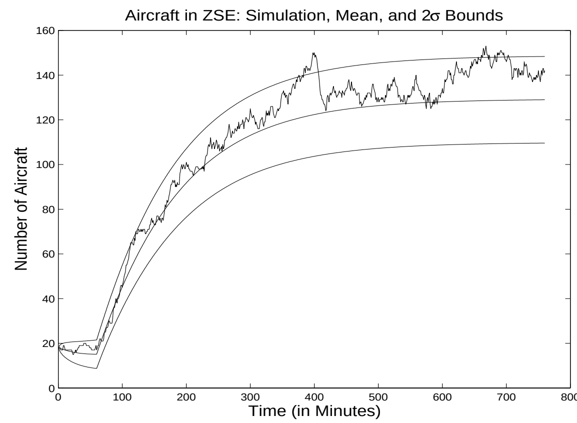

Fig. 6. This plot shows a simulation of a flow model for the U.S. ATS. In particular, the number of aircraft in ZSE is simulated at one-minute intervals, over a duration of 760 minutes. In addition to the flow model simulation of the number of aircraft in ZSE, the expected number of aircraft in ZSE, conditioned on the initial counts of all Centers, and 2-standard deviation bands around this expected number are plotted. The initial Center counts in this simulation are based on actual data of Center counts at approximately 5:00 AM PDT on September 6th, 2000. We have assumed that there are no departures from airports in ZSE until 6:00 AM, and then departures commence at a nominal daytime rate, explaining the sudden jump in the aircraft count at 6:00 AM.

In addition to the conditional mean of each Center count, higher moments and cross moments of the numbers of aircraft in each Center (conditioned on an initial state) can be computed through a linear recursive procedure. We have developed the recursion for the second-order moments and cross-moments. The analysis of the second-order moments is straightforward, but the resulting expression for the moments at each time-step is not very illustrative. Thus, we only briefly describe the analysis in the Appendix. In Figure 6, the computation of the second moments is used to develop 2σ bounds on the number of aircraft in ZSE.

C. Verification of the Model

First, we would like to show that a Poisson process description for the ATS is adequate. Our methodology is founded on the assumption that departures from airports are described accurately by Poisson processes. At the time scales of interest to us (on the order of a few minutes to a few hours), the data that we have explored suggests that departures from large airports are indeed essentially Poisson in nature. For example, we plot a histogram of the number of daytime departures during one-minute intervals at Chicago O'Hare Airport (ORD) in Figure 7, and find that the departures in a minute are well modeled by a Poisson random variable. Also, we find that the number of aircraft departing in any given minute is independent of the number of aircraft departing at other times. Both these properties suggest that the departure process is well represented as Poisson, at a one-minute granularity. We have found similar behavior at other airports, and for time-scales of greater than one minute. At smaller airports, a Poisson process description with a fixed mean is not accurate, but we believe that a time-varying Poisson process description could be reasonable; alternately, the aggregate departure process from several small airports can perhaps be modeled as a Poisson process. At any rate, the departures from large airports contribute most significantly to air traffic, so we believe that a Poisson process description for departures is justified. To further investigate our Poisson process framework, the distribution of the number of aircraft in a Center is also studied. For example, an empirical distribution of the number of aircraft in the Seattle Center (ZSE) is compared with a Poisson distribution of the same mean in Figure 8. The variances of the empirical and fit distributions are close (117.6 and 127.6, respectively), and the two distributions are similar in shape. The statistics of other Centers are similar. Thus, we believe that a Poisson random variable model is a good description for the number of aircraft in a Center.

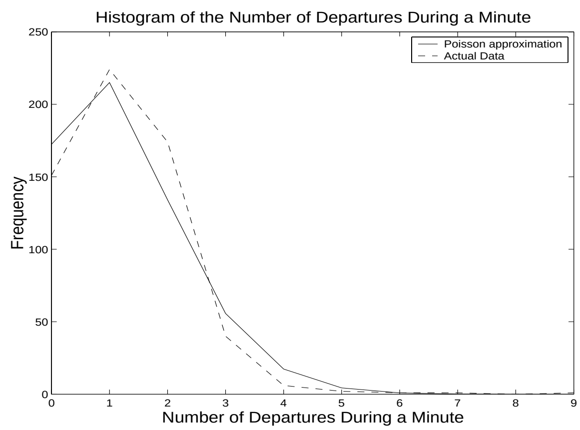

Fig. 7. An empirical distribution (histogram) of the number of aircraft departing from an airport (ORD) during minute intervals is shown. The empirical distribution is based on data from ORD over 600 consecutive minutes; over this time interval, the average departure rate from the airport remained approximately constant. Also, the empirical distribution is compared with a Poisson distribution with the same mean. The comparison suggests that the number of aircraft departing from ORD in minute intervals are indeed well-represented by a Poisson random variable.

Second, we explore whether our model, which is motivated by the Poisson process description of an ATS but is not identical to it, can accurately represent the dynamics of the U.S. ATS. We can check whether or not our model is consistent with a Poisson process description of the ATS. For example, the standard deviation in the steady-state aircraft count predicted by the model can be compared with the standard deviation if the count were modeled by a Poisson random variable of the same mean (in accordance with the Poisson process description of the ATS). In our example, the standard deviation for the number of aircraft in ZSE predicted by our model is 9.7, while the standard deviation predicted by a Poisson random variable representation is 11.3. In general, we find that the standard deviations for Center counts predicted by our model are slightly smaller than, but close to, the standard deviations predicted by the Poisson process description of an ATS.

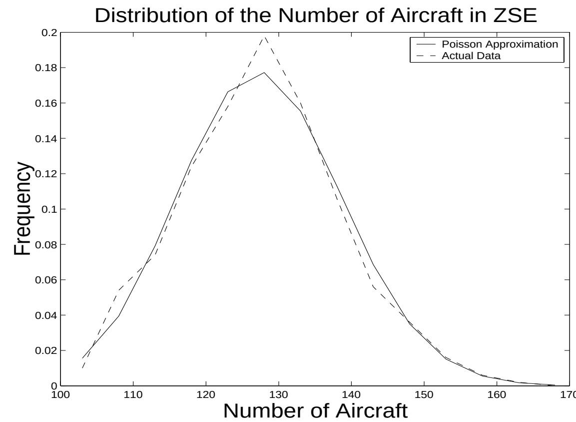

Fig. 8. An empirical distribution for the number of aircraft in ZSE is shown, and is compared with a Poisson random variable of the same mean. The empirical distribution is generated based on 500 observations at one-minute intervals. The sample variance of the empirical distribution and the variance of the Poisson approximation are similar.

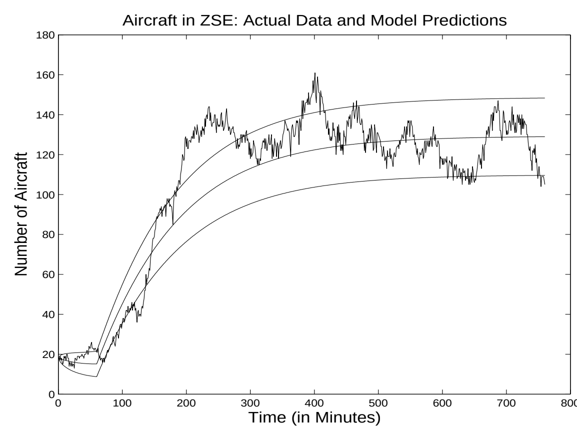

Fig. 9. The actual number of aircraft in ZSE at 760 consecutive one-minute time steps is compared with the mean number of aircraft predicted by our model, using the actual Center counts at all Centers at the time of the first data point (5:00 AM PDT, September 6th, 2000) as the initial conditions for the model. In our model, we have assumed that aircraft departures in ZSE begin at 6:00 AM. This plot also includes the 2σ bounds on the aircraft count in ZSE predicted by our model. The actual data is largely contained within two standard deviations of the predicted mean, suggesting that the model predictions for the mean and standard deviation are both reasonable. The one noticeable difference between the actual data and the model prediction is the sluggishness in the model's transient as compared to the data.

Another approach for verifying the model is to check whether or not the behavior of the model matches actual Center count data. In Figure 9, the number of aircraft in ZSE during a 12-hour period on September 6th, 2000 is traced. Furthermore, the model prediction for mean and standard deviation for the number of aircraft in ZSE, given the initial numbers of aircraft in all Centers, is plotted for comparison; these are the same as in Figure 6. The actual data largely falls within two standard deviations of the predicted mean, suggesting that the model provides a good description of the dynamics of the ATS. The most noticeable difference between the model prediction and the actual data is the slower settling time to the steady-state of the prediction. Based on our explorations, we believe that this sluggishness in the model comes about because aircraft in the model do not spend fixed amounts of time in Centers, and instead may be delayed for lengthy time periods with some (small) probability.

In our comparison of the model prediction with historical data, we chose to track the Center counts beginning early in the morning of September 6th, even though the parameters of the model are based only on afternoon and evening Center counts. It is reassuring that the model can predict the Center counts from this initial condition, albeit imperfectly. After all, the primary goal of our methodology is to develop a *dynamic* model—one which can track the transient behavior of the ATS (such as the increase in traffic in the airspace in the morning), as well as the spatial and temporal correlations in Center counts.

IV. Extensions of the Basic Model

A. Extension 1: A Hierarchical Model

Real ATS naturally have a hierarchical structure: for example, Centers in the U.S. ATS are further subdivided into smaller regions, called *Sectors*. Modeling of parts of the U.S. ATS at a Sector level is often valuable, both because many air traffic control and flow management decisions are made at the Sector level, and because a model with a finer spatial granularity can predict the future behavior of the system more accurately. A model for Sector counts also may require a finer time-sampling, because of the occurrence of significant dynamics over shorter time intervals. On the other hand, parts of the ATS can sometimes be aggregated in a model, reducing the computational complexity of the analysis while still correctly representing the parts of the system of interest.

Our modeling methodology is well-suited for describing a system at various spatial and temporal granularities. For example, consider an ATS with three levels of representation (again, we adopt the terminology for the U.S. ATS in our model):

1. Some regions in the airspace are modeled at a Sector level. Thus, the number of aircraft in each Sector in these regions is represented. These Sector counts are tracked at intervals of ΔT.
2. Some regions in the airspace are modeled at a Center level. The Center counts in these regions are tracked less often, at intervals of $f\Delta T$, for some positive integer f.
3. Some regions of the airspace are not modeled at all, and aircraft counts in these regions are not tracked (though aircraft flows to and from these regions are still incorporated in the model).

Figure 10 shows such a hierarchical model for the ATS.

A system with these three levels of representation can be modeled in our framework, as follows:

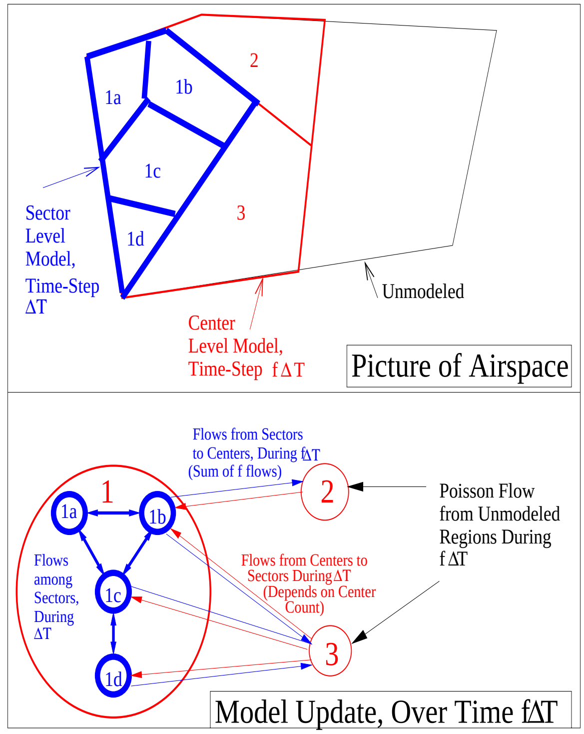

Fig. 10. This figure shows a hierarchical model for the ATS which incorporates dynamics at various levels of aggregation and time-scales. The upper portion of the figure is a drawing of the airspace, while the lower portion describes the equivalent model.

- First consider the update of Sector counts, in regions where Sector-level dynamics are represented. Sector counts are updated at intervals of ΔT. Each of these Sectors has flows into and out of other parts of the airspace. In general, a Sector may have flows to and from other modeled Sectors, regions modeled at the Center level, unmodeled regions of the airspace, and the exterior of the system (i.e., flows to and from airports within the Sector). The flows out of a Sector in an interval ΔT are computed based on probabilities that aircraft flow to each other region, and out of the airspace (which can be found by deducing the rates of traffic flow among the appropriate regions of the airspace using historical data). Meanwhile, flows into a Sector are generated in different ways in the simulation. Flows from the exterior of the system, and from unmodeled regions, are modeled as Poisson processes. Flows from regions modeled at a Center level are determined based on the most recently updated counts in these Centers (along with flow probabilities), and flows from other modeled Sectors are determined based on these Sector's counts at the previous time-step along with flow probabilities.
- Aircraft counts in regions modeled at the Center level are updated at intervals of $f\Delta T$. The flows out of each Center to other Centers, unmodeled regions of the airspace, and the exterior of the system, are determined based on the current count of the Center and probabilities of aircraft going to other regions of the airspace. The flows out of a Center to modeled Sectors are found by summing the appropriate flows into the Sector (which have already been

generated) over f intervals of duration ΔT. Next, consider flows into these Centers. Flows from the exterior of the airspace and from unmodeled regions are generated as Poisson processes. Flows from other Centers are generated as in the basic model, using flow probabilities and counts in these Centers. Finally, the flows from modeled Sectors to Centers are found by summing the appropriate flows out of these Sectors over f time-steps.

B. Extension 2: A Model with Stochastic, Flow-Altering Disturbances

The U.S. ATS is subject to disturbances that change rates of aircraft flow in parts of the network. Many of these flow-altering disturbances, which are often inclement weather events in parts of the airspace, cannot accurately be predicted in advance. Furthermore, although the disturbance event may directly affect only a small part of the airspace, the resulting changes in flows and Sector/Center counts may propagate throughout the network. Since our model for the U.S. ATS is stochastic, we can naturally incorporate stochastic disturbances that alter flows in the model. By computing the expected behavior and variability of Center counts and flows in the model, regions of the airspace that may be prone to capacity excesses due to the weather events can be identified. In turn, the model may suggest improved methods for managing traffic flow in response to weather disturbances.

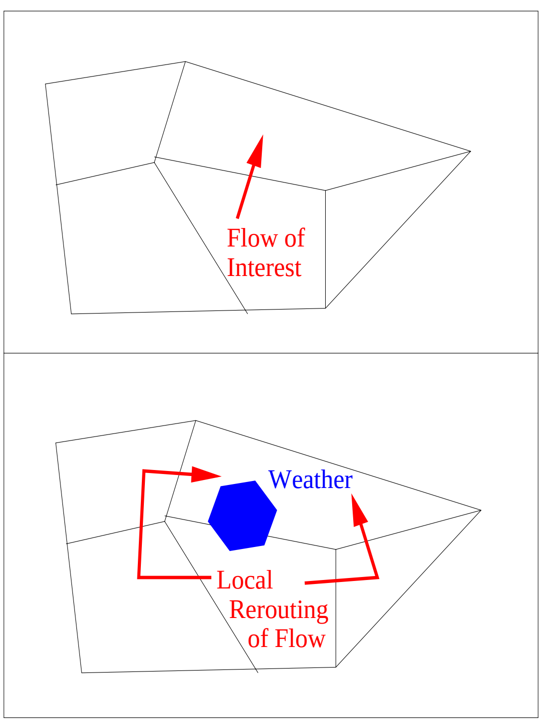

Fig. 11. Local perturbation of flows due to a weather event is depicted. In this case, the flow of interested is rerouted in two different directions, leading to different flow rates across each boundary.

We model local perturbations as changes to the nominal parameters, as shown in Figure 11. In our approach, multiple disturbances, each of which occur independently with some probability, can affect flows in a ATS. Given that a particular set of disturbances has occurred, we can calculate statistics of Center counts with our basic model, using the appropriate set of model parameters (which are modified from their nominal values based on the particular disturbances that have occurred). In turn, we can calculate statistics of Center counts without prior knowledge of the disturbances, by scaling the predicted statistics for each set of disturbances with the probability that these disturbances occur, and then summing these scaled statistics. In this way, the dynamics of an ATS that is subject to stochastic disturbances can be modeled and analyzed.

One possible shortcoming of this approach for modeling stochastic disturbances is the computational complexity resulting from the large number of disturbances that may need to be considered. (For example, if there are 10 different weather events that may or may not be present on a given day, we must consider $2^{10} = 1024$ possible combinations of disturbances.) Given certain special conditions on the location of disturbances, the computational complexity can sometimes be reduced by considering the change in the system's dynamics due to each disturbance separately, and then combining these individual responses.

V. Conclusions and Future Work

In this article, we have proposed an aggregate stochastic model for an ATS. Some analyses of the model that may eventually prove useful for predicting and controlling flows in an ATS have been discussed. Also, some verification of our modeling framework has been attempted, and two extensions of the basic model have been outlined.

We believe that aggregate stochastic models provide a promising description of the ATS, but more study is required to gauge the value of these models in improving understanding and operation of the ATS. To better undertand the benefits and drawbacks of using aggregate and stochastic models, we plan to compare our model with other deterministic and detailed stochastic models for the ATS. We are also interested in using the model as a framework for developing Traffic Flow Management (TFM) algorithms.

Appendix

Equation 2 shows how the expected numbers of aircraft in each Center at each time-step can be calculated recursively, given the initial numbers of aircraft in all Centers. A detailed proof for Equation 2 will be given elsewhere [12]. In this appendix, we explain how the recursion in Equation 2 comes about, without concerning ourselves with the vector notation of the recursion. To do so, consider $E(s_i[k+1]\,|\,\mathbf{s}[k]$, the expected number of aircraft in a Center i at time $k+1$, given all Center counts at time k. This expectation can be found by adding and subtracting the expected flows into and out of Center i, respectively, from $s_i[k]$. Note that the expected number of aircraft that flow from Center i to a Center j (or out of the system for $j=0$) is given by $s_i[k]p_{ij}[k]$, the number that flow from a Center j to Center i is $s_j[k]p_{ji}[k]$, and the number that depart from airports in Center i is $\lambda_{0i}[k]$. Thus, the conditional expectation for the number of aircraft in Center i is

$$E(s_i[k+1]\,|\,\mathbf{s}[k]) \tag{3}$$
$$= s_i[k] - \sum_{\substack{j=0\\j\neq i}}^{n} s_i[k]p_{ij}[k] + (\sum_{\substack{j=1\\j\neq i}}^{n} s_j[k]p_{ji}[k] + \lambda_i[k]),$$

which is a linear function of the time-k Center counts. Finally, by taking the expectation of Equation 3 with respect to the time-k Center counts $\mathbf{s}[k]$, given the initial Center counts $\mathbf{s}[0]$, we find that

$$E(s_i[k+1]\,|\,\mathbf{s}[0]) \tag{4}$$
$$= E(s_i[k]\,|\mathbf{s}[0]) - \sum_{\substack{j=0\\j\neq i}}^{n} E(s_i[k]\,|\,\mathbf{s}[0])p_{ij}[k]$$
$$+(\sum_{\substack{j=1\\j\neq i}}^{n} E(s_j[k]\,|\,\mathbf{s}[0])p_{ji}[k] + \lambda_i[k]).$$

Thus, we see that the expected number of aircraft in Center i at time $k+1$ given $\mathbf{s}[0]$ can be written as a linear function of the expected Center counts at time k given $\mathbf{s}[0]$.

In this appendix, we also briefly discuss why second moments and cross-moments of state variables can be found using linear recursions given the initial state $\mathbf{s}[0]$. A detailed exposition of the second, and higher, moment calculations is given in [12]. The recursion for the second moments and cross-moments is derived analogously to the recursion for the expectations of state variables. For example, consider $E(s_i^2[k+1]\,|\,\mathbf{s}[k])$, the expected value of the square of the number of aircraft in a Center i at time $k+1$ given the Center counts at time k. Substituting the expression for $s_i[k+1]$ given in Equation 1 into this expectation gives

$$E(s_i^2[k+1]\,|\,\mathbf{s}[k]) \tag{5}$$
$$= E((s_i[k] - \sum_{\substack{j=0\\j\neq i}}^{n} U_{ij}[k] + \sum_{\substack{j=0\\j\neq i}}^{n} U_{ji}[k])^2\,|\,\mathbf{s}[k])$$

To continue the analysis, note that $\sum_{\substack{j=0\\j\neq i}}^{n} U_{ij}[k]$ and $\sum_{\substack{j=0\\j\neq i}}^{n} U_{ji}[k]$ are two independent Poisson random variables given the time-k Center counts $\mathbf{s}[k]$, with means $\sum_{\substack{j=0\\j\neq i}}^{n} s_i[k]p_{ij}[k]$ and $\sum_{\substack{j=1\\j\neq i}}^{n} s_j[k]p_{ji}[k] + \lambda_i[k]$, respectively. Since the second moment of a Poisson random variable is a quadratic function of its mean, we find after some algebra that the expectation $E(s_i^2[k+1]\,|\,\mathbf{s}[k])$ is a quadratic function of the state variables at time k (i.e., the expectation can be written as a sum of second- and lower-order powers of the state variables, such as $s_i^2[k]$, $s_i[k]s_j[k]$, $s_i[k]$, etc.). Taking the expectation of Equation 5 with respect to $\mathbf{s}[k]$, we find that the second moment of $s_i[k+1]$ (given $\mathbf{s}[0]$) is a linear function of second and lower moments and cross-moments of the time-k state variables (also given $\mathbf{s}[0]$). In fact, it turns out that all second moments and cross-moments of time-$(k+1)$ state variables can be written as linear functions of second and lower moments and cross-moments of time-k state variables. Thus, the second-order statistics of the model can be found using linear recursions.

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